# XX

# Algebraic Geometry II: Affine Schemes

# Part XX: Contents

# 82 Sheaves and ringed spaces

Most of the complexity of the affine variety $V$ earlier comes from $\mathcal{O}_V$. This is a type of object called a “sheaf”. The purpose of this chapter is to completely define what this sheaf is, and just what it is doing.

## §82.1 Motivation and warnings

The typical example to keep in mind is a sheaf of “functions with property $P$” on a topological space $X$: for every open set $U$, $\mathscr{F}(U)$ gives us the ring of functions on $X$. However, we will work very abstractly and only assume $\mathscr{F}(U)$ is a ring, without an interpretation as “functions”.

Throughout this chapter, I will not only be using algebraic geometry examples, but also those with $X$ a topological space and $\mathscr{F}$ being a sheaf of differentiable/analytic/etc functions. One of the nice things about sheaves is that the same abstraction works fine, so you can train your intuition with both algebraic and analytic examples. In particular, we can keep drawing open sets $U$ as ovals, even though in the Zariski topology that's not what they look like.

The payoff for this abstraction is that it will allow us to define an arbitrary scheme in Chapter 84. Varieties use $\mathbb{C}[x_1, x_2, \dots, x_n]/I$ as their “ring of functions”, and by using the fully general sheaf we replace this with *any* commutative ring. In particular, we could choose $\mathbb{C}[x]/(x^2)$ and this will give the “multiplicity” behavior that we sought all along.

## §82.2 Pre-sheaves

*Prototypical example for this section: The sheaf of holomorphic (or regular, continuous, differentiable, constant, whatever) functions.*

The proper generalization of our $\mathcal{O}_V$ is a so-called sheaf of rings. Recall that $\mathcal{O}_V$ took *open sets of* $V$ to *rings*, with the interpretation that $\mathcal{O}_V(U)$ was a “ring of functions”.

Recall from Section 78.5 that $\mathcal{O}_V$, as a set, consist of simply the algebraic functions. However, if we view $\mathcal{O}_V$ purely as a set, the structure of the functions is essentially thrown way.

Let us see how the functions in $\mathcal{O}_V$ are related to each other:

- Each function in $\mathcal{O}_V$ is defined on a open set $U \subseteq V$.
- If two functions are defined on the same open set, you can add and multiply them together. In other words, $\mathcal{O}_V(U)$ is a ring.
- Given a function $f \in \mathcal{O}_V(U)$, we can restrict it to a smaller open subset $W \subseteq U$.

These are the operations that we will impose on a pre-sheaf.

### §82.2.i Usual definition

So here is the official definition of a pre-sheaf. We will only define a pre-sheaf of rings, however it's possible to define a pre-sheaf of sets, pre-sheaf of abelian groups, etc.

**Definition 82.2.1.** For a topological space $X$ let $\operatorname{OpenSets}(X)$ denote the open sets of $X$.

**Definition 82.2.2.** A **pre-sheaf** of rings on a space $X$ is a function

$$\mathscr{F} \colon \operatorname{OpenSets}(X) \to \mathsf{Rings}$$

meaning each open set gets associated with a ring $\mathscr{F}(U)$. Each individual element of $\mathscr{F}(U)$ is called a **section**.

It is also equipped with a **restriction map** for any $U_1 \subseteq U_2$; this is a map

$$\operatorname{res}_{U_1,U_2} \colon \mathscr{F}(U_2) \to \mathscr{F}(U_1).$$

The map satisfies two axioms:

- The map $\operatorname{res}_{U,U}$ is the identity, and
- Whenever we have nested subsets

  $$U_{\text{small}} \subseteq U_{\text{med}} \subseteq U_{\text{big}}$$

  the diagram

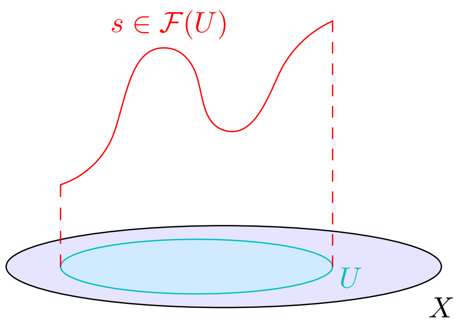
  $$\begin{tikzcd} \mathscr{F}(U_{\text{big}}) \arrow[r, "\text{res}"] \arrow[dr, "\text{res}"'] & \mathscr{F}(U_{\text{med}}) \arrow[d, "\text{res}"] \\ & \mathscr{F}(U_{\text{small}}) \end{tikzcd}$$

  commutes.

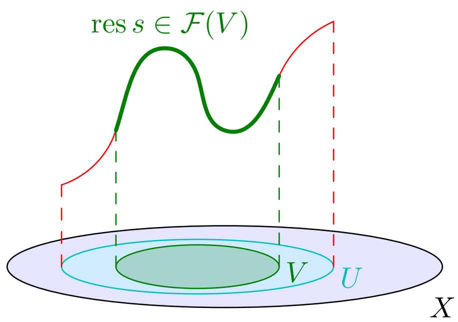
**Definition 82.2.3.** An element of $\mathscr{F}(X)$ is called a **global section**.

**Abuse of Notation 82.2.4.** If $s \in \mathscr{F}(U_2)$ is some section and $U_1 \subseteq U_2$, then rather than write $\operatorname{res}_{U_1,U_2}(s)$ I will write $s\restriction_{U_1}$ instead: "$s$ restricted to $U_1$". This is abuse of notation because the section $s$ is just an element of some ring, and in the most abstract of cases may not have a natural interpretation as function.

Here is a way you can picture sections. In all our usual examples, sheaves return functions an open set $U$. So, we draw a space $X$, and an open set $U$, and then we want to draw a "function on $U$" to represent a section $s$. Crudely, we will illustrate $s$ by drawing an $xy$-plot of a curve, since that is how we were told to draw functions in grade school.




Then, the restriction corresponds to, well, taking just a chunk of the section.




All of this is still a dream, since $s$ in reality is an element of a ring. However, by the end of this chapter we will be able to make our dream into a reality.

**Example 82.2.5** (Examples of pre-sheaves)

(a) For an affine variety $V$, $\mathcal{O}_V$ is of course a sheaf, with $\mathcal{O}_V(U)$ being the ring of regular functions on $U$. The restriction map just says that if $U_1 \subseteq U_2$, then a function $s \in \mathcal{O}_V(U_2)$ can also be thought of as a function $s\restriction_{U_1} \in \mathcal{O}_V(U_1)$, hence the name "restriction". The commutativity of the diagram then follows.

(b) Let $X \subseteq \mathbb{R}^n$ be an open set. Then there is a sheaf of smooth/differentiable/etc. functions on $X$. In fact, one can do the same construction for any manifold $M$.

(c) Similarly, if $X \subseteq \mathbb{C}$ is open, we can construct a sheaf of holomorphic functions on $X$.

In all these examples, the sections $s \in \mathscr{F}(U)$ are really functions on the space, but in general they need not be.

In practice, thinking about the restriction maps might be more confusing than helpful; it is better to say:

**Pre-sheaves should be thought of as "returning the ring of functions with a property $P$".**

## §82.2.ii Categorical definition

If you really like category theory, we can give a second equivalent and shorter definition. Despite being a category lover myself, I find this definition less intuitive, but its brevity helps with remembering the first one.

**Abuse of Notation 82.2.6.** By abuse of notation, $\operatorname{OpenSets}(X)$ will also be thought of as a posetal category by inclusion. Thus $\varnothing$ is an initial object and the entire space $X$ is a terminal object.

**Definition 82.2.7.** A **pre-sheaf** of rings on $X$ is a contravariant functor

$$\mathscr{F} \colon \operatorname{OpenSets}(X)^{\mathrm{op}} \to \mathsf{Rings}.$$

**Exercise 82.2.8.** Check that these definitions are equivalent.

In particular, it is possible to replace $\mathsf{Rings}$ with any category we want. We will not need to do so any time soon, but it's worth mentioning.

## §82.3 Stalks and germs

*Prototypical example for this section: Germs of real smooth functions tell you the derivatives, but germs of holomorphic functions determine the entire function.*

As we mentioned, the helpful pictures from the previous section are still just metaphors, because there is no notion of "value". With the addition of the words "stalk" and "germ", we can actually change that.

**Definition 82.3.1.** Let $\mathscr{F}$ be a pre-sheaf (of rings). For every point $p$ we define the **stalk** $\mathscr{F}_p$ to be the set

$$\{(s, U) \mid s \in \mathscr{F}(U), p \in U\}$$

modulo the equivalence relation $\sim$ that

$$(s_1, U_1) \sim (s_2, U_2) \quad \text{if} \quad s_1\restriction_V = s_2\restriction_V$$

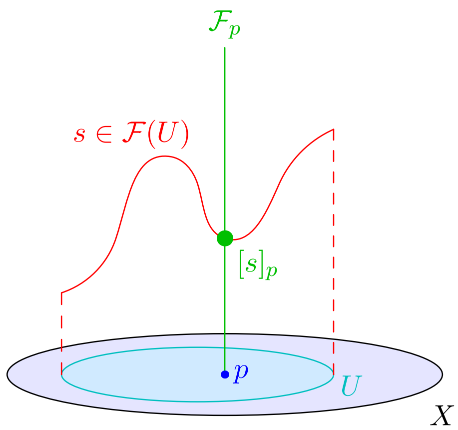
for some open set $V$ with $V \ni p$ and $V \subseteq U_1 \cap U_2$. The equivalence classes themselves are called **germs**.

**Definition 82.3.2.** The germ of a given $s \in \mathscr{F}(U)$ at a point $p$ is the equivalence class for $(s, U) \in \mathscr{F}_p$. We denote this by $[s]_p$.

It is rarely useful to think of a germ as an ordered pair, since the set $U$ can get arbitrarily small. Instead, one should think of a germ as a "shred" of some section near $p$. A nice summary for the right mindset might be:

**A germ is an "enriched value"; the stalk is the set of possible germs.**

Let's add this to our picture from before. If we insist on continuing to draw our sections as $xy$-plots, then above each point $p$ a good metaphor would be a vertical line out from $p$. The germ would then be the "enriched value of $s$ at $p$". We just draw that as a big dot in our plot. The main difference is that the germ is enriched in the sense that the germ carries information in a small region around $p$ as well, rather than literally just the point $p$ itself. So accordingly we draw a large dot for $[s]_p$, rather than a small dot at $p$.




Before going on, we might as well note that the stalks are themselves rings, not just sets: we can certainly add or subtract enriched values.

**Definition 82.3.3.** The stalk $\mathscr{F}_p$ can itself be regarded as a ring: for example, addition is done by

$$(s_1, U_1) + (s_2, U_2) = (s_1\restriction_{U_1 \cap U_2} + s_2\restriction_{U_1 \cap U_2}, U_1 \cap U_2)\,.$$

**Example 82.3.4** (Germs of real smooth functions)

Let $X = \mathbb{R}$ and let $\mathscr{F}$ be the pre-sheaf on $X$ of smooth functions (i.e. $\mathscr{F}(U)$ is the set of smooth real-valued functions on $U$).
Consider a global section, $s\colon \mathbb{R} \to \mathbb{R}$ (thus $s \in \mathscr{F}(X)$) and its germ at 0.

(a) From the germ we can read off $s(0)$, obviously.

(b) We can also find $s'(0)$, because the germ carries enough information to compute the limit $\lim_{h\to 0} \frac{1}{h}[s(h) - s(0)]$.

(c) Similarly, we can compute the second derivative and so on.

(d) However, we can't read off, say, $s(3)$ from the germ. For example, consider the function from Example 29.4.4,

$$s(x) = \begin{cases} e^{-\frac{1}{x-1}} & x > 1 \\ 0 & x \le 1. \end{cases}$$

Note $s(3) = e^{-\frac{1}{2}}$, but $[\text{zero function}]_0 = [s]_0$. So germs can't distinguish between the zero function and $s$.

**Example 82.3.5** (Germs of holomorphic functions)

Holomorphic functions are surprising in this respect. Consider the sheaf $\mathscr{F}$ on $\mathbb{C}$ of *holomorphic* functions.
Take $s\colon \mathbb{C} \to \mathbb{C}$ a global section. Given the germ of $s$ at 0, we can read off $s(0)$, $s'(0)$, et cetera. The miracle of complex analysis is that just knowing the derivatives of $s$ at zero is enough to reconstruct all of $s$: we can compute the Taylor series of $s$ now. **Thus germs of holomorphic functions determine the entire function**; they "carry more information" than their real counterparts.
In particular, we can concretely describe the stalks of the pre-sheaf:

$$\mathscr{F}_p = \left\{ \sum_{k\ge 0} c_k(z-p)^k \text{ convergent near } p \right\}.$$

For example, this includes germs of meromorphic functions, so long as there is no pole at $p$ itself.

And of course, our algebraic geometry example. This example will matter a lot later, so we do it carefully now.

**Abuse of Notation 82.3.6.** Rather than writing $(\mathcal{O}_X)_p$ we will write $\mathcal{O}_{X,p}$.

**Theorem 82.3.7** (Stalks of $\mathcal{O}_V$)

Let $V \subseteq \mathbb{A}^n$ be a variety, and assume $p \in V$ is a point. Then

$$\mathcal{O}_{V,p} \cong \left\{ \frac{f}{g} \mid f, g \in \mathbb{C}[V],\ g(p) \neq 0 \right\}.$$

*Proof.* A regular function $\varphi$ on $U \subseteq V$ is supposed to be a function on $U$ that "locally" is a quotient of two functions in $\mathbb{C}[V]$. Since we are looking at the stalk near $p$, though, the germ only cares up to the choice of representation at $p$, and so we can go ahead and write

$$\mathcal{O}_{V,p} = \left\{ \left( \tfrac{f}{g}, U \right) \mid U \ni p, \; f, g \in \mathbb{C}[V], \; g \neq 0 \text{ on } U \right\}$$

modulo the same relation.

Now we claim that the map

$$\mathcal{O}_{V,p} \to \text{desired RHS} \qquad \text{by} \qquad \left( \frac{f}{g}, U \right) \mapsto \frac{f}{g}$$

is an isomorphism.

- Injectivity: We are working with complex polynomials, so we know that a rational function is determined by its behavior on any open neighborhood of $p$; thus two germ representatives $(\frac{f_1}{g_1}, U_1)$ and $(\frac{f_2}{g_2}, U_2)$ agree on $U_1 \cap U_2$ if and only if they are actually the same quotient.
- Surjectivity: take $U = D(g)$. $\square$

**Example 82.3.8** (Stalks of your favorite varieties at the origin)

(a) Let $V = \mathbb{A}^1$; then the stalk of $\mathcal{O}_V$ at each point $p \in V$ is

$$\mathcal{O}_{V,p} = \left\{ \frac{f(x)}{g(x)} \mid g(p) \neq 0 \right\}.$$

Examples of elements are $x^2 + 5$, $\frac{1}{x-1}$ if $p \neq 1$, $\frac{x+7}{x^2-9}$ if $p \neq \pm 3$, and so on.

(b) Let $V = \mathbb{A}^2$; then the stalk of $\mathcal{O}_V$ at the origin is

$$\mathcal{O}_{V,(0,0)} = \left\{ \frac{f(x,y)}{g(x,y)} \mid g(0,0) \neq 0 \right\}.$$

Examples of elements are $x^2 + y^2$, $\frac{x^3}{xy+1}$, $\frac{13x+37y}{x^2+8y+2}$.

(c) Let $V = \mathcal{V}(y - x^2) \subseteq \mathbb{A}^2$; then the stalk of $\mathcal{O}_V$ at the origin is

$$\mathcal{O}_{V,(0,0)} = \left\{ \frac{f(x,y)}{g(x,y)} \mid f, g \in \mathbb{C}[x,y]/(y - x^2), \; g(0,0) \neq 0 \right\}.$$

For example, $\frac{y}{1+x} = \frac{x^2}{1+x}$ denote the same element in the stalk. Actually, you could give a canonical choice of representative by replacing $y$ with $x^2$ everywhere, so it would also be correct to write

$$\mathcal{O}_{V,(0,0)} = \left\{ \frac{f(x)}{g(x)} \mid \; g(0) \neq 0 \right\}$$

which is the same as the first example.

**Remark 82.3.9** (Aside for category lovers) — You may notice that $\mathscr{F}_p$ seems to be "all the $\mathscr{F}_p(U)$ coming together", where $p \in U$. And in fact, $\mathscr{F}_p(U)$ is the categorical *colimit* of the diagram formed by all the $\mathscr{F}(U)$ such that $p \in U$. This is often written

$$\mathscr{F}_p = \varinjlim_{U \ni p} \mathscr{F}(U)$$

Thus we can define stalks in any category with colimits, though to be able to talk about germs the category needs to be concrete.

## §82.4 Sheaves

*Prototypical example for this section: Constant functions aren't sheaves, but locally constant ones are.*

Since we care so much about stalks, which study local behavior, we will impose additional local conditions on our pre-sheaves. One way to think about this is:

**Sheaves are pre-sheaves for which $P$ is a *local* property.**

The formal definition doesn't illuminate this as much as the examples do, but sadly I have to give the definition first for the examples to make sense.

**Definition 82.4.1.** A **sheaf** $\mathscr{F}$ on a topological space $X$ is a pre-sheaf obeying two additional axioms: Suppose $U$ is an open set in $X$, and $U$ is covered by open sets $U_\alpha \subseteq U$. Then:

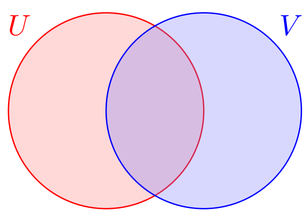
1. (Identity) If $s, t \in \mathscr{F}(U)$ are sections, and $s \restriction_{U_\alpha} = t \restriction_{U_\alpha}$ for all $\alpha$, then $s = t$.
2. (Gluing) Consider sections $s_\alpha \in \mathscr{F}(U_\alpha)$ for each $\alpha$. Suppose that

$$s_\alpha \restriction_{U_\alpha \cap U_\beta} = s_\beta \restriction_{U_\alpha \cap U_\beta}$$

for each $U_\alpha$ and $U_\beta$. Then we can find $s \in \mathscr{F}(U)$ such that $s \restriction_{U_\alpha} = s_\alpha$.

**Remark 82.4.2** (For keepers of the empty set) — The above axioms imply $\mathscr{F}(\varnothing) = 0$ (the zero ring), when $\mathscr{F}$ is a sheaf of rings. This is not worth worrying about until you actually need it, so you can forget I said that.

This is best illustrated by picture in the case of just two open sets: consider two open sets $U$ and $V$. Then the sheaf axioms are saying something about $\mathscr{F}(U \cup V)$, $\mathscr{F}(U \cap V)$, $\mathscr{F}(U)$ and $\mathscr{F}(V)$.




Then for a sheaf of functions, the axioms are saying that:

- If $s$ and $t$ are functions (with property $P$) on $U \cup V$ and $s \restriction_U = t \restriction_U$, $s \restriction_V = t \restriction_V$, then $s = t$ on the entire union. This is clear.

- If $s_1$ is a function with property $P$ on $U$ and $s_2$ is a function with property $P$ on $V$, and the two functions agree on the overlap, then one can glue them to obtain a function $s$ on the whole space: this is obvious, but **the catch is that the collated function needs to have property $P$ as well** (i.e. needs to be an element of $\mathscr{F}(U \cup V)$). That's why it matters that $P$ is local.

So you can summarize both of these as saying: any two functions on $U$ and $V$ which agree on the overlap glue to a *unique* function on $U \cup V$. If you like category theory, you might remember we alluded to this in Example 69.2.1.

**Exercise 82.4.3** (For the categorically inclined)**.** Show that the diagram

$$\begin{tikzcd} \mathscr{F}(U \cup V) \arrow[r] \arrow[d] & \mathscr{F}(U) \arrow[d] \\ \mathscr{F}(V) \arrow[r] & \mathscr{F}(U \cap V) \end{tikzcd}$$

is a pullback square.

Now for the examples.

**Example 82.4.4** (Examples and non-examples of sheaves)

Note that every example of a stalk we computed in the previous section was of a sheaf. Here are more details:

(a) Pre-sheaves of arbitrary / continuous / differentiable / smooth / holomorphic functions are still sheaves. This is because to verify a function is continuous, one only needs to look at small open neighborhoods at once.

(b) Let $X = \mathbb{R}$, and define the presheaf of rings $\mathscr{F}$ by:

$$\mathscr{F}(U) = \{f : U \to \mathbb{R} \mid \text{there exists continuous } g : \mathbb{R} \to \mathbb{R} \text{ such that } g\restriction_U = f\}.$$

Then $\mathscr{F}$ is not a sheaf. Indeed, $s_1(x) = 0$ in $\mathscr{F}((-1,0))$ and $s_2(x) = 1$ in $\mathscr{F}((0,1))$ agrees on the (empty) overlap, but they cannot be glued together to an element in $\mathscr{F}((-1,0) \cup (0,1))$.

(c) For a complex variety $V$, $\mathcal{O}_V$ is a sheaf, precisely because our definition was *locally* quotients of polynomials.

(d) The pre-sheaf of *constant* real functions on a space $X$ is *not* a sheaf in general, because it fails the gluing axiom. Namely, suppose that $U_1 \cap U_2 = \varnothing$ are disjoint open sets of $X$. Then if $s_1$ is the constant function 1 on $U_1$ while $s_2$ is the constant function 2 on $U_2$, then we cannot glue these to a constant function on $U_1 \cup U_2$.

(e) On the other hand, *locally constant* functions do produce a sheaf. (A function is locally constant if for every point it is constant on some open neighborhood.)

In fact, the sheaf in e is what is called a *sheafification* of the pre-sheaf constant functions, which we define momentarily.

## §82.5 For sheaves, sections "are" sequences of germs

*Prototypical example for this section: A real function on $U$ is a sequence of real numbers $f(p)$ for each $p \in U$ satisfying some local condition. Analogously, a section $s \in \mathscr{F}(U)$ is a sequence of germs satisfying some local compatibility condition.*

Once we impose the sheaf axioms, our metaphorical picture will actually be more or less complete. Just as a function was supposed to be a choice of value at each point, a section will be a choice of germ at each stalk.

**Example 82.5.1** (Real functions vs. germs)
Let $X$ be a space and let $\mathscr{F}$ be the sheaf of smooth functions. Take a section $f \in \mathscr{F}(U)$.

- As a function, $f$ is just a choice of value $f(p) \in \mathbb{R}$ at every point $p$, subject to a local "smooth" condition.
- Let's now think of $f$ as a sequence of germs. At every point $p$ the germ $[f]_p \in \mathscr{F}_p$ gives us the value $f(p)$ as we described above. The germ packages even more data than this: from the germ $[f]_p$ alone we can for example compute $f'(p)$. Nonetheless we stretch the analogy and think of $f$ as a choice of germ $[f]_p \in \mathscr{F}_p$ at each point $p$.

Thus we can replace the notion of the value $f(p)$ with germ $[f]_p$. This is useful because in a general sheaf $\mathscr{F}$, the notion $s(p)$ is not defined while the notion $[s]_p$ is.

From the above example it's obvious that if we know each germ $[s]_p$, this should let us reconstruct the entire section $s$. Let's check this from the sheaf axioms:

**Exercise 82.5.2** (Sections are determined by stalks)**.** Let $\mathscr{F}$ be a sheaf. Consider the natural map

$$\mathscr{F}(U) \to \prod_{p \in U} \mathscr{F}_p$$

described above. Show that this map is injective, i.e. the germs of $s$ at every point $p \in U$ determine the section $s$. (You will need the "identity" sheaf axiom, but not "gluing".)

However, this map is clearly not surjective! Nonetheless we can describe the image: we want a sequence of germs $(g_p)_{p\in U}$ such that near every germ $g_p$, the germs $g_q$ are "compatible" with $g_p$. We make this precise:

**Definition 82.5.3.** Let $\mathscr{F}$ be pre-sheaf and let $U$ be an open set. A sequence $(g_p)_{p\in U}$ of germs (with $g_p \in \mathscr{F}_p$ for each $p$) is said to be **compatible** if they can be "locally collated":

> For any $p \in U$ there exists an open neighborhood $U_p \ni p$ and a section $s \in \mathscr{F}(U_p)$ on it such that $[s]_q = g_q$ for each $q \in U_p$.

Intuitively, the germs should "collate together" to some section near each *individual* point $q$ (but not necessarily to a section on all of $U$).

We let the reader check this definition is what we want:

**Exercise 82.5.4.** Prove that any choice of compatible germs over $U$ collates together to a section of $U$. (You will need the "gluing" sheaf axiom, but not "identity".)

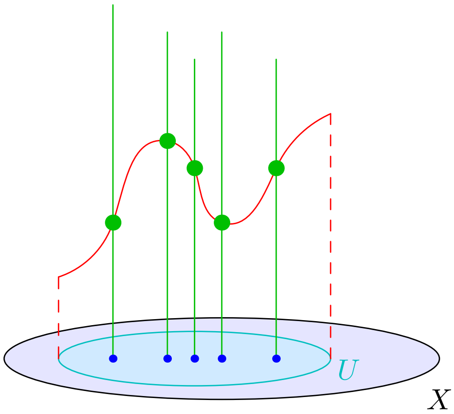
Putting together the previous two exercise gives:

**Theorem 82.5.5** (Sections "are" just compatible germs)

Let $\mathscr{F}$ be a sheaf. There is a natural bijection between

- sections of $\mathscr{F}(U)$, and
- sequences of compatible germs over $U$.

We draw this in a picture below by drawing several stalks, rather than just one, with the germs above. The stalks at different points need not be related to each other, so I have drawn the stalks to have different heights to signal this. And, the caveat is that the germs are large enough that germs which are "close to each other" should be "compatible".




This is in exact analogy to the way that e.g. a smooth real-valued function on $U$ is a choice of real number $f(p) \in \mathbb{R}$ at each point $p \in U$ satisfying a local smoothness condition.

Thus the notion of stalks is what lets us recover the viewpoint that sections are "functions". Therefore for theoretical purposes,

**With sheaf axioms, sections are sequences of compatible germs.**

In particular, this makes restriction morphisms easy to deal with: just truncate the sequence of germs!

## §82.6 Sheafification (optional)

*Prototypical example for this section: The pre-sheaf of constant functions becomes the sheaf of locally constant functions.*

The idea is that if $\mathscr{F}$ is the pre-sheaf of "functions with property $P$" then we want to associate a sheaf $\mathscr{F}^{\text{sh}}$ of "functions which are locally $P$", which makes them into a sheaf. We have already seen two examples of this:

**Example 82.6.1** (Sheafification)

(a) If $X$ is a topological space, and $\mathscr{F}$ is the pre-sheaf of constant functions on open sets of $X$, then $\mathscr{F}^{\text{sh}}$ is the sheaf of locally constant functions.

(b) If $V$ is an affine variety, and $\mathscr{F}$ is the pre-sheaf of rational functions, then $\mathscr{F}^{\text{sh}}$ is the sheaf of regular functions (which are locally rational).

The procedure is based on stalks and germs. We saw that for a sheaf, sections correspond to sequences of compatible germs. For a pre-sheaf, we can still define stalks and germs, but their properties will be less nice. But given our initial pre-sheaf $\mathscr{F}$, we *define* the sections of $\mathscr{F}^{\text{sh}}$ to be sequences of compatible $\mathscr{F}$-germs.

**Definition 82.6.2.** The **sheafification** $\mathscr{F}^{\text{sh}}$ of a pre-sheaf $\mathscr{F}$ is defined by

$$\mathscr{F}^{\text{sh}}(U) = \{\text{sequences of compatible } \mathscr{F}\text{-germs } (g_p)_{p\in U}\}.$$

**Question 82.6.3.** Complete the definition by describing the restriction morphisms of $\mathscr{F}^{\text{sh}}$.

**Abuse of Notation 82.6.4.** I'll usually be equally sloppy in the future: when defining a sheaf $\mathscr{F}$, I'll only say what $\mathscr{F}(U)$ is, with the restriction morphisms $\mathscr{F}(U_2) \to \mathscr{F}(U_1)$ being implicit.

The construction is contrived so that given a section $(g_p)_{p\in U} \in \mathscr{F}^{\text{sh}}(U)$ the germ at a point $p$ is $g_p$:

**Lemma 82.6.5** (Stalks preserved by sheafification)

Let $\mathscr{F}$ be a pre-sheaf and $\mathscr{F}^{\text{sh}}$ its sheafification. Then for any point $q$, there is an isomorphism

$$(\mathscr{F}^{\text{sh}})_q \cong \mathscr{F}_q.$$

*Proof.* A germ in $(\mathscr{F}^{\text{sh}})_q$ looks like $((g_p)_{p\in U}, U)$, where $g_p = (s_p, U_p)$ are themselves germs of $\mathscr{F}_p$, and $q \in U$. Then the isomorphism is given by

$$((g_p)_{p\in U}, U) \mapsto g_q \in \mathscr{F}_q.$$

The inverse map is given by for each $g = (s, U) \in \mathscr{F}_q$ by

$$g \mapsto ((g)_{p\in U}, U) \in (\mathscr{F}^{\text{sh}})_q$$

i.e. the sequence of germs is the constant sequence. □

We will use sheafification in the future to economically construct sheaves. However, in practice, the details of the construction will often not matter.

## §82.7 A few harder problems to think about

**Problem 82A.** Prove that if $\mathscr{F}$ is already a sheaf, then $\mathscr{F}(U) \cong \mathscr{F}^{\text{sh}}(U)$ for every open set $U$.

**Problem 82B.** Let $X$ be a space with two points $\{p, q\}$ and let $\mathscr{F}$ be a sheaf on it. Suppose $\mathscr{F}_p = \mathbb{Z}/5\mathbb{Z}$ and $\mathscr{F}_q = \mathbb{Z}$. Describe $\mathscr{F}(U)$ for each open set $U$ of $X$, where

(a) $X$ is equipped with the discrete topology.

(b) $X$ is equipped $\varnothing$, $\{p\}$, $\{p, q\}$ as the only open sets.

**Problem 82C** (Skyscraper sheaf)**.** Let $Y$ be a topological space. Fix $p \in Y$ a point, and $R$ a ring. The **skyscraper sheaf** is defined by

$$\mathscr{F}(U) = \begin{cases} R & p \in U \\ 0 & \text{otherwise} \end{cases}$$

with restriction maps in the obvious manner. Compute all the stalks of $\mathscr{F}$.

(Possible suggestion: first do the case where $Y$ is Hausdorff, where your intuition will give the right answer. Then do the pathological case where every open set of $Y$ contains $p$. Then try to work out the general answer.)

**Problem 82D.** Let $\mathscr{F}$ be a sheaf of rings on a space $X$ and let $s \in \mathscr{F}(X)$ be a global section. Define the **support** of $s$ as

$$Z = \{p \in X \mid [s]_p \neq 0 \in \mathscr{F}_p\}.$$

Show that $Z$ is a closed set of $X$.

# 83 Localization

Before we proceed on to defining an affine scheme, we will take the time to properly cover one more algebraic construction that of a *localization*. This is mandatory because when we define a scheme, we will find that all the sections and stalks are actually obtained using this construction.

One silly slogan might be:

**Localization is the art of adding denominators.**

You may remember that when we were working with affine varieties, there were constantly expressions of the form $\left\{\frac{f}{g} \mid g(p) \neq 0\right\}$ and the like. The point is that we introduced a lot of denominators. Localization will give us a concise way of doing this in general.

This thus also explain why the operation is called "localization": we start from a set of "global" functions, and get a (larger) set of functions that are well-defined on a "smaller" open set, or in an open neighborhood of point $p$.

Of course, this is the Zariski topology, so "small" means "everywhere except certain curves".

*Notational note*: moving forward we'll prefer to denote rings by $A$, $B$, ..., rather than $R$, $S$, ....

## §83.1 Spoilers

Here is a preview of things to come, so that you know what you are expecting. Some things here won't make sense, but that's okay, it is just foreshadowing.

Let $V \subseteq \mathbb{A}^n$, and for brevity let $R = \mathbb{C}[V]$ be its coordinate ring. We saw in previous sections how to compute $\mathcal{O}_V(D(g))$ and $\mathcal{O}_{V,p}$ for $p \in V$ a point. For example, if we take $\mathbb{A}^1$ and consider a point $p$, then $\mathcal{O}_{\mathbb{A}^1}(D(x-p)) = \left\{\frac{f(x)}{(x-p)^n}\right\}$ and $\mathcal{O}_{\mathbb{A}^1,p} = \left\{\frac{f(x)}{g(x)} \mid g(p) \neq 0\right\}$. More generally, we had

$$\mathcal{O}_V(D(g)) = \left\{\frac{f}{g^n} \mid f \in R\right\} \quad \text{by Theorem 78.6.1}$$
$$\mathcal{O}_{V,p} = \left\{\frac{f}{g} \mid f, g \in R, g(p) \neq 0\right\} \quad \text{by Theorem 82.3.7.}$$

We will soon define something called a localization, which will give us a nice way of expressing the above: if $R = \mathbb{C}[V]$ is the coordinate ring, then the above will become abbreviated to just

$$\mathcal{O}_V(D(g)) = R[g^{-1}]$$
$$\mathcal{O}_{V,p} = R_{\mathfrak{m}} \quad \text{where } \{p\} = \mathcal{V}(\mathfrak{m}).$$

The former will be pronounced "$R$ localized away from $g$" while the latter will be pronounced "$R$ localized at $\mathfrak{m}$".

Even more generally, next chapter we will throw out the coordinate ring $R$ altogether and replace it with a general commutative ring $A$ (which are still viewed as functions).

We will construct a ringed space called $X = \operatorname{Spec} A$, whose elements are *prime ideals* of $A$ and is equipped with the Zariski topology and a sheaf $\mathcal{O}_X$. It will turn out that, the right way to define the sheaf $\mathcal{O}_X$ is to use localization,

$$\mathcal{O}_X(D(f)) = A[f^{-1}]$$
$$\mathcal{O}_{X,\mathfrak{p}} = A_{\mathfrak{p}}$$

for any element $f \in A$ and prime ideal $\mathfrak{p} \in \operatorname{Spec} A$. Thus just as with complex affine varieties, localizations will give us a way to more or less describe the sheaf $\mathcal{O}_X$ completely.

In other words,

**Localization is the purely algebraic way to *define* the ring of regular functions on a smaller open set from the ring of "global" regular functions.**

## §83.2 The definition

**Definition 83.2.1.** A subset $S \subseteq A$ is a **multiplicative set** if $1 \in S$ and $S$ is closed under multiplication.

**Definition 83.2.2.** Let $A$ be a ring and $S \subset A$ a multiplicative set. Then the **localization of $A$ at $S$**, denoted $S^{-1}A$, is defined as the set of fractions

$$\{a/s \mid a \in A, s \in S\}$$

where we declare two fractions $a_1/s_1 = a_2/s_2$ to be equal if $s(a_1s_2 - a_2s_1) = 0$ for some $s \in S$. Addition and multiplication in this ring are defined in the obvious way.

In particular, if $0 \in S$ then $S^{-1}A$ is the zero ring. So we usually only take situations where $0 \notin S$.

We give in brief now two examples which will be motivating forces for the construction of the affine scheme.

**Example 83.2.3** (Localizations of $\mathbb{C}[x]$)

Let $A = \mathbb{C}[x]$.

(a) Suppose we let $S = \{1, x, x^2, x^3, \dots\}$ be the powers of $x$. Then

$$S^{-1}A = \left\{ \frac{f(x)}{x^n} \mid f \in \mathbb{C}[x], n \in \mathbb{Z}_{\ge 0} \right\}.$$

In other words, we get the Laurent polynomials in $x$.

You might recognize this as

$$\mathcal{O}_V(U) \text{ where } V = \mathbb{A}^1,\ U = V \setminus \{0\}.$$

i.e. the sections of the punctured line. In line with the "hyperbola effect", this is also expressible as $\mathbb{C}[x,y]/(xy-1)$.

(b) Let $p \in \mathbb{C}$. Suppose we let $S = \{g(x) \mid g(p) \neq 0\}$, i.e. we allow any denominators where $g(p) \neq 0$. Then

$$S^{-1}A = \left\{ \frac{f(x)}{g(x)} \mid f, g \in \mathbb{C}[x], g(p) \neq 0 \right\}.$$

You might recognize this is as the stalk $\mathcal{O}_{\mathbb{A}^1,p}$. This will be important later on.

**Remark 83.2.4** (Why the extra $s$?) — We cannot use the simpler $a_1s_2 - a_2s_1 = 0$ since otherwise the equivalence relation may fail to be transitive. Here is a counterexample: take

$$A = \mathbb{Z}/12\mathbb{Z} \qquad S = \{2, 4, 8\}.$$

Then we have for example $\frac{0}{1} = \frac{0}{2} = \frac{12}{2} = \frac{6}{1}$. So we need to have $\frac{0}{1} = \frac{6}{1}$ which is only true with the first definition. Of course, if $A$ is an integral domain (and $0 \notin S$) then this is a moot point.
Alternatively, one can start with this simpler relation, and take the transitive closure; this yields an equivalent definition.

**Example 83.2.5** (Field of fractions)
Let $A$ be an integral domain and $S = A \setminus \{0\}$. Then $S^{-1}A = \operatorname{Frac}(A)$.

## §83.3 Localization away from an element

*Prototypical example for this section:* $\mathbb{Z}$ *localized away from* $6$ *has fractions* $\frac{m}{2^x3^y}$.

We now focus on the two special cases of localization we will need the most; one in this section, the other in the next section.

**Definition 83.3.1.** For $f \in A$, we define the **localization of $A$ away from $f$**, denoted $A[1/f]$ or $A[f^{-1}]$, to be $\{1, f, f^2, f^3, \dots\}^{-1}A$. (Note that $\{1, f, f^2, \dots\}$ is multiplicative.)

**Remark 83.3.2** — In the literature it is more common to see the notation $A_f$ instead of $A[1/f]$. This is confusing, because in the next section we define $A_{\mathfrak{p}}$ which is almost the opposite. So I prefer this more suggestive (but longer) notation.

**Example 83.3.3** (Some arithmetic examples of localizations)

(a) We localize $\mathbb{Z}$ away from 6:

$$\mathbb{Z}[1/6] = \left\{ \frac{m}{6^n} \mid m \in \mathbb{Z}, n \in \mathbb{Z}_{\geq 0} \right\}.$$

So $A[1/6]$ consist of those rational numbers whose denominators have only powers of 2 and 3. For example, it contains $\frac{5}{12} = \frac{15}{36}$.

(b) Here is a more confusing example: if we localize $\mathbb{Z}/60\mathbb{Z}$ away from the element 5, we get $(\mathbb{Z}/60\mathbb{Z})[1/5] \cong \mathbb{Z}/12\mathbb{Z}$. You should try to think about why this is the case. We will see a "geometric" reason later, in Section 86.9.

**Example 83.3.4** (Localization at an element, algebraic geometry flavored)
We saw that if $A$ is the coordinate ring of a variety, then $A[1/g]$ is interpreted geometrically as $\mathcal{O}_V(D(g))$. Here are some special cases:

(a) As we saw, if $A = \mathbb{C}[x]$, then $A[1/x] = \left\{ \frac{f(x)}{x^n} \right\}$ consists of Laurent polynomials.

(b) Let $A = \mathbb{C}[x, y, z]$. Then

$$A[1/x] = \left\{ \frac{f(x,y,z)}{x^n} \mid f \in \mathbb{C}[x,y,z],\ n \geq 0 \right\}$$

is rational functions whose denominators are powers of $x$.

(c) Let $A = \mathbb{C}[x, y]$. If we localize away from $y - x^2$ we get

$$A[(y-x^2)^{-1}] = \left\{ \frac{f(x,y)}{(y-x^2)^n} \mid f \in \mathbb{C}[x,y],\ n \geq 0 \right\}$$

By now you should recognize this as $\mathcal{O}_{\mathbb{A}^2}(D(y - x^2))$.

**Example 83.3.5** (An example with zero-divisors)

Let $A = \mathbb{C}[x, y]/(xy)$ (which intuitively is the coordinate ring of two axes). Suppose we localize at $x$: equivalently, allowing denominators of $x$. Since $xy = 0$ in $A$, we now have $0 = x^{-1}(xy) = y$, so $y = 0$ in $A$, and thus $y$ just goes away completely. From this we get a ring isomorphism

$$A[1/x] \cong \mathbb{C}[x, 1/x].$$

Later, we will be able to use our geometric intuition to "see" this at Section 86.14, once we have defined the affine scheme.

## §83.4 Localization at a prime ideal

*Prototypical example for this section: $\mathbb{Z}$ localized at $(5)$ has fractions $\frac{m}{n}$ with $5 \nmid n$.*

**Definition 83.4.1.** If $A$ is a ring and $\mathfrak{p}$ is a prime ideal, then we define

$$A_{\mathfrak{p}} := (A \setminus \mathfrak{p})^{-1} A.$$

This is called the **localization at $\mathfrak{p}$**.

**Question 83.4.2.** Why is $S = A \setminus \mathfrak{p}$ multiplicative in the above definition?

This special case is important because we will see that stalks of schemes will all be of this shape. In fact, the same was true for affine varieties too.

**Example 83.4.3** (Relation to affine varieties)

Let $V \subseteq \mathbb{A}^n$, let $A = \mathbb{C}[V]$ and let $p = (a_1, \dots, a_n)$ be a point. Consider the maximal (hence prime) ideal

$$\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n).$$

Observe that a function $f \in A$ vanishes at $p$ if and only if $f \pmod{\mathfrak{m}} = 0$, equivalently $f \in \mathfrak{m}$. Thus, by Theorem 82.3.7 we can write

$$\begin{aligned} \mathcal{O}_{V,p} &= \left\{ \frac{f}{g} \mid f, g \in A, g(p) \neq 0 \right\} \\ &= \left\{ \frac{f}{g} \mid f \in A, g \in A \setminus \mathfrak{m} \right\} \\ &= (A \setminus \mathfrak{m})^{-1} A = A_{\mathfrak{m}}. \end{aligned}$$

So, we can also express $\mathcal{O}_{V,p}$ concisely as a localization.

Consequently, we give several examples in this vein.

**Example 83.4.4** (Geometric examples of localizing at a prime)

(a) We let $\mathfrak{m}$ be the maximal ideal $(x)$ of $A = \mathbb{C}[x]$. Then

$$A_{\mathfrak{m}} = \left\{ \frac{f(x)}{g(x)} \mid g(0) \neq 0 \right\}$$

consists of the Laurent series.

(b) We let $\mathfrak{m}$ be the maximal ideal $(x, y)$ of $A = \mathbb{C}[x, y]$. Then

$$A_{\mathfrak{m}} = \left\{ \frac{f(x,y)}{g(x,y)} \mid g(0,0) \neq 0 \right\}.$$

(c) Let $\mathfrak{p}$ be the prime ideal $(y - x^2)$ of $A = \mathbb{C}[x, y]$. Then

$$A_{\mathfrak{p}} = \left\{ \frac{f(x,y)}{g(x,y)} \mid g \notin (y - x^2) \right\}.$$

This is a bit different from what we've seen before: the polynomials in the denominator are allowed to vanish at a point like $(1, 1)$, as long as they don't vanish on *every* point on the parabola. This doesn't correspond to any stalk we're familiar with right now, but it will later (it will be the "stalk at the generic point of the parabola").

(d) Let $A = \mathbb{C}[x]$ and localize at the prime ideal $(0)$. This gives

$$A_{(0)} = \left\{ \frac{f(x)}{g(x)} \mid g(x) \neq 0 \right\}.$$

This is all rational functions, period.

**Remark 83.4.5** (Notational philosophy) To reiterate:

- when localizing away from an element, you allow the functions to blow up at (the vanishing set of) that element;
- when localizing at a prime, you allow the functions to blow up everywhere *except* at (the whole vanishing set of) that prime.

Thus we see why we say the 2 notations are opposites.
Thinking of functions that "may not blow up at the whole vanishing set" can be confusing, so another (hopefully) more intuitive way to think about localizing at a prime is that the function must not blow up at the point corresponding to the prime ideal. For example, if $\mathfrak{p}$ is the ideal $(y - x^2)$ in $A = \mathbb{C}[x, y]$, then $A_\mathfrak{p}$ is the set of functions that do not blow up at the generic point on the parabola.

**Example 83.4.6** (Arithmetic examples)
We localize $\mathbb{Z}$ at a few different primes.

(a) If we localize $\mathbb{Z}$ at $(0)$:
$$\mathbb{Z}_{(0)} = \left\{ \frac{m}{n} \mid n \neq 0 \right\} \cong \mathbb{Q}.$$

(b) If we localize $\mathbb{Z}$ at $(3)$, we get
$$\mathbb{Z}_{(3)} = \left\{ \frac{m}{n} \mid \gcd(n, 3) = 1 \right\}$$
which is the ring of rational numbers whose denominators are relatively prime to 3.

**Example 83.4.7** (Field of fractions)
If $A$ is an integral domain, the localization $A_{(0)}$ is the field of fractions of $A$.

## §83.5 Prime ideals of localizations

*Prototypical example for this section: The examples with $A = \mathbb{Z}$.*

We take the time now to mention how you can think about prime ideals of localized rings.

**Proposition 83.5.1** (The prime ideals of $S^{-1}A$)
Let $A$ be a ring and $S \subseteq A$ a multiplicative set. Then there is a natural inclusion-preserving bijection between:

- The set of prime ideals of $S^{-1}A$, and
- The set of prime ideals of $A$ not intersecting $S$.

*Proof.* Consider the homomorphism $\iota \colon A \to S^{-1}A$. For any prime ideal $\mathfrak{q} \subseteq S^{-1}A$, its pre-image $\iota^{\text{pre}}(\mathfrak{q})$ is a prime ideal of $A$ (by Problem 5C$^\star$). Conversely, for any prime ideal $\mathfrak{p} \subseteq A$ not meeting $S$, $S^{-1}\mathfrak{p} = \left\{ \frac{a}{s} \mid a \in \mathfrak{p}, s \in S \right\}$ is a prime ideal of $S^{-1}A$. An annoying check shows that this produces the required bijection. □

In practice, we will almost always use the corollary where $S$ is one of the two special cases we discussed at length:

**Corollary 83.5.2** (Spectrums of localizations)

Let $A$ be a ring.

(a) If $f$ is an element of $A$, then the prime ideals of $A[1/f]$ are naturally in bijection with prime ideals of $A$ **do not contain the element** $f$.

(b) If $\mathfrak{p}$ is a prime ideal of $A$, then the prime ideals of $A_\mathfrak{p}$ are naturally in bijection with prime ideals of $A$ which are **subsets of** $\mathfrak{p}$.

*Proof.* Part (b) is immediate; a prime ideal doesn't meet $A \setminus \mathfrak{p}$ exactly if it is contained in $\mathfrak{p}$. For part (a), we want prime ideals of $A$ not containing any *power* of $f$. But if the ideal is prime and contains $f^n$, then it should contain either $f$ or $f^{n-1}$, and so at least for prime ideals these are equivalent. □

Notice again how the notation is a bit of a nuisance. Anyways, here are some examples, to help cement the picture.

**Example 83.5.3** (Prime ideals of $\mathbb{Z}[1/6]$)

Suppose we localize $\mathbb{Z}$ away from the element 6, i.e. consider $\mathbb{Z}[1/6]$. As we saw,

$$\mathbb{Z}[1/6] = \left\{ \frac{n}{2^x 3^y} \mid n \in \mathbb{Z}, x, y \in \mathbb{Z}_{\geq 0} \right\}.$$

consist of those rational numbers whose denominators have only powers of 2 and 3. Note that $(5) \subset \mathbb{Z}[1/6]$ is a prime ideal: those elements of $\mathbb{Z}[1/6]$ with 5 dividing the numerator. Similarly, $(7)$, $(11)$, $(13)$, ... and even $(0)$ give prime ideals of $\mathbb{Z}[1/6]$. But $(2)$ and $(3)$ no longer correspond to prime ideals; in fact in $\mathbb{Z}[1/6]$ we have $(2) = (3) = (1)$, the whole ring.

**Example 83.5.4** (Prime ideals of $\mathbb{Z}_{(5)}$)

Suppose we localize $\mathbb{Z}$ at the prime $(5)$. As we saw,

$$\mathbb{Z}_{(5)} = \left\{ \frac{m}{n} \mid m, n \in \mathbb{Z}, 5 \nmid n \right\}.$$

consist of those rational numbers whose denominators are not divisible by 5. This is an integral domain, so $(0)$ is still a prime ideal. There is one other prime ideal: $(5)$, i.e. those elements whose numerators are divisible by 5.
There are no other prime ideals: if $p \neq 5$ is a rational prime, then $(p) = (1)$, the whole ring, again.

## §83.6 Prime ideals of quotients

While we are here, we mention that the prime ideals of quotients $A/I$ can be interpreted in terms of those of $A$ (as in the previous section for localization). You may remember this from Problem 4D$^\star$ a long time ago, if you did that problem; but for our purposes we actually only care about the prime ideals.

**Proposition 83.6.1** (The prime ideals of $A/I$)

If $A$ is a ring and $I$ is any ideal (not necessarily prime) then the prime (resp. maximal) ideals of $A/I$ are in bijection with prime (resp. maximal) ideals of $A$ which are **supersets of** $I$. This bijection is inclusion-preserving.

*Proof.* Consider the quotient homomorphism $\psi \colon A \twoheadrightarrow A/I$. For any prime ideal $\mathfrak{q} \subseteq A/I$, its pre-image $\psi^{\text{pre}}(\mathfrak{q})$ is a prime ideal (by Problem 5C*). Conversely, for any prime ideal $\mathfrak{p}$ with $I \subseteq \mathfrak{p} \subseteq A$, we get a prime ideal of $A/I$ by looking at $\mathfrak{p} \pmod I$. An annoying check shows that this produces the required bijection. It is also inclusion-preserving — from which the same statement holds for maximal ideals. □

**Example 83.6.2** (Prime ideals of $\mathbb{Z}/60\mathbb{Z}$)

The ring $\mathbb{Z}/60\mathbb{Z}$ has three prime ideals:

$$\begin{aligned}(2) &= \{0, 2, 4, \dots, 58\} \\ (3) &= \{0, 3, 6, \dots, 57\} \\ (5) &= \{0, 5, 10, \dots, 55\}.\end{aligned}$$

Back in $\mathbb{Z}$, these correspond to the three prime ideals which are supersets of $60\mathbb{Z} = \{\dots, -60, 0, 60, 120, \dots\}$.

## §83.7 Localization commutes with quotients

*Prototypical example for this section:* $(\mathbb{C}[x,y]/(xy))[1/x] \cong \mathbb{C}[x, x^{-1}]$.

While we are here, we mention a useful result from commutative algebra which lets us compute localizations in quotient rings, which are surprisingly unintuitive. You will *not* have a reason to care about this until we reach Section 85.4.ii, and so this is only placed earlier to emphasize that it's a purely algebraic fact that we can (and do) state this early, even though we will not need it anytime soon.

Let's say we have a quotient ring like

$$A/I = \mathbb{C}[x,y]/(xy)$$

and want to compute the localization of this ring away from the element $x$. (To be pedantic, we are actually localizing away from $x \pmod{xy}$, the element of the quotient ring, but we will just call it $x$.) You will quickly find that even the notation becomes clumsy: it is

$$\left(\mathbb{C}[x,y]/(xy)\right)[1/x] \tag{83.1}$$

which is hard to think about, because the elements in play are part of the *quotient*: how are we supposed to think about

$$\frac{1 \pmod{xy}}{x \pmod{xy}}$$

for example? The zero-divisors in play may already make you feel uneasy.

However, it turns out that we can actually do the localization *first*, meaning the answer is just

$$\mathbb{C}[x, y, 1/x]/(xy) \tag{83.2}$$

which then becomes $\mathbb{C}[x, x^{-1}, y]/(y) \cong \mathbb{C}[x, x^{-1}]$.

This might look like it should be trivial, but it's not as obvious as you might expect. There is a sleight of hand present here with the notation:

- In (83.1), the notation $(xy)$ stands for an ideal of $\mathbb{C}[x,y]$ — that is, the set $xy\mathbb{C}[x,y]$.
- In (83.2) the notation $(xy)$ now stands for an ideal of $\mathbb{C}[x,x^{-1},y]$ — that is, the set $xy\mathbb{C}[x,x^{-1},y]$.

So even writing down the *statement* of the theorem is actually going to look terrible.

In general, what we want to say is that if we have our ring $A$ with ideal $I$ and $S$ is some multiplicative subset of $A$, then

$$\text{Colloquially: “}S^{-1}(A/I) = (S^{-1}A)/I\text{”.}$$

But there are two things wrong with this:

- The main one is that $I$ is not an ideal of $S^{-1}A$, as we saw above. This is remedied by instead using $S^{-1}I$, which consists of those elements of those elements $\frac{x}{s}$ for $x \in I$ and $s \in S$. As we saw this distinction is usually masked in practice, because we will usually write $I = (a_1, \dots, a_n) \subseteq A$ in which case the new ideal $S^{-1}I \subseteq A$ can be denoted in exactly the same way: $(a_1, \dots, a_n)$, just regarded as a subset of $S^{-1}A$ now.
- The second is that $S$ is not, strictly speaking, a subset of $A/I$, either. But this is easily remedied by instead using the image of $S$ under the quotient map $A \twoheadrightarrow A/I$.

  We actually already saw this in the previous example: when trying to localize $\mathbb{C}[x,y]/(xy)$, we were really localizing at the element $x \pmod{xy}$, but (as always) we just denoted it by $x$ anyways.

And so after all those words, words, words, we have the hideous:

**Theorem 83.7.1** (Localization commutes with quotients)
Let $S$ be a multiplicative set of a ring $A$, and $I$ an ideal of $A$. Let $\overline{S}$ be the image of $S$ under the projection map $A \twoheadrightarrow A/I$. Then

$$\overline{S}^{-1}(A/I) \cong S^{-1}A/S^{-1}I$$

where $S^{-1}I = \{\frac{x}{s} \mid x \in I, s \in S\}$.

*Proof.* Omitted; Atiyah-Macdonald is the right reference for these type of things in the event that you do care. $\square$

The notation is a hot mess. But when we do calculations in practice, we instead write

$$\left(\mathbb{C}[x,y,z]/(x^2+y^2-z^2)\right)[1/x] \cong \mathbb{C}[x,y,z,1/x]/(x^2+y^2-z^2)$$

or (for an example where we localize at a prime ideal)

$$\left(\mathbb{Z}[x,y,z]/(x^2+yz)\right)_{(x,y)} \cong \mathbb{Z}[x,y,z]_{(x,y)}/(x^2+yz)$$

and so on — the pragmatism of our “real-life” notation which hides some details actually guides our intuition (rather than misleading us). So maybe the moral of this section is that whenever you compute the localization of the quotient ring, if you just suspend belief for a bit, then you will probably get the right answer.

We will later see geometric interpretations of these facts when we work with $\operatorname{Spec} A/I$, at which point they will become more natural.

## §83.8 A few harder problems to think about

**Problem 83A.** Let $A = \mathbb{Z}/2016\mathbb{Z}$, and consider the element $60 \in A$. Compute $A[1/60]$, the localization of $A$ away from 60.

**Problem 83B** (Injectivity of localizations)**.** Let $A$ be a ring and $S \subseteq A$ a multiplicative set. Find necessary and sufficient conditions for the map $A \to S^{-1}A$ to be injective.

**Problem 83C**$^\star$ (Alluding to local rings)**.** Let $A$ be a ring, and $\mathfrak{p}$ a prime ideal. How many maximal ideals does $A_{\mathfrak{p}}$ have?

**Problem 83D.** Let $A$ be a nonzero ring such that $A_{\mathfrak{p}}$ is an integral domain for every prime ideal $\mathfrak{p}$ of $A$. Must $A$ be an integral domain?

# 84 Affine schemes: the Zariski topology

Now that we understand sheaves well, we can define an affine scheme. It will be a ringed space, so we need to define

- The set of points,
- The topology on it, and
- The structure sheaf on it.

In this chapter, we handle the first two parts; Chapter 85 does the last one.

Quick note: Chapter 86 contains a long list of examples of affine schemes. So if something written in this chapter is not making sense, one thing worth trying is skimming through Chapter 86 to see if any of the examples there are more helpful.

## §84.1 Some more advertising

Let me describe what the construction of $\operatorname{Spec} A$ is going to do.

In the case of $\mathbb{A}^n$, we used $\mathbb{C}^n$ as the set of points and $\mathbb{C}[x_1, \dots, x_n]$ as the ring of functions but then remarked that the set of points of $\mathbb{C}^n$ corresponded to the maximal ideals of $\mathbb{C}[x_1, \dots, x_n]$. In an *affine scheme*, we will take an *arbitrary* ring $A$, and generate the entire structure from just $A$ itself. The final result is called $\operatorname{Spec} A$, the **spectrum** of $A$. The affine varieties $\mathcal{V}(I)$ we met earlier will just be $\operatorname{Spec} \mathbb{C}[\mathcal{V}(I)] = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]/I$ , but now we will be able to take *any* ideal $I$, thus finally completing the table at the end of the "affine variety" chapter.

To emphasize the point:

**For affine varieties $V$, the spectrum of the coordinate ring $\mathbb{C}[V]$ is $V$.**

Thus, we may also think of Spec as the opposite operation of taking the ring of global sections, defined purely-algebraically in order to depend only on the intrinsic properties of the affine variety itself (the ring $\mathcal{O}_V$) and not the embedding.

The construction of the affine scheme in this way will have three big generalizations:

1. We no longer have to work over an algebraically closed field $\mathbb{C}$, or even a field at all. This will be the most painless generalization: you won't have to adjust your current picture much for this to work.

2. We allow non-radical ideals: $\operatorname{Spec} \mathbb{C}[x]/(x^2)$ will be the double point we sought for so long. This will let us formalize the notion of a "fat" or "fuzzy" point.

3. Our affine schemes will have so-called *non-closed points*: points which you can visualize as floating around, somewhere in the space but nowhere in particular. (They'll correspond to prime non-maximal ideals.) These will take the longest to get used to, but as we progress we will begin to see that these non-closed points actually make life *easier*, once you get a sense of what they look like.

## §84.2 The set of points

*Prototypical example for this section:* $\operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]/I$.

First surprise, for a ring $A$:

**Definition 84.2.1.** The set $\operatorname{Spec} A$ is defined as the set of prime ideals of $A$.

This might be a little surprising, since we might have guessed that $\operatorname{Spec} A$ should just have the maximal ideals. What do the remaining ideals correspond to? The answer is that they will be so-called *non-closed points* or *generic points* which are "somewhere" in the space, but nowhere in particular. (The name "non-closed" is explained next chapter.)

**Remark 84.2.2** — As usual $A$ itself is not a prime ideal, but $(0)$ is prime if and only if $A$ is an integral domain.

**Example 84.2.3** (Examples of spectrums)

(a) $\operatorname{Spec} \mathbb{C}[x]$ consists of a point $(x-a)$ for every $a \in \mathbb{C}$, which correspond to what we geometrically think of as $\mathbb{A}^1$. It additionally consists of a point $(0)$, which we think of as a "non-closed point", nowhere in particular.

(b) $\operatorname{Spec} \mathbb{C}[x, y]$ consists of points $(x-a, y-b)$ (which are the maximal ideals) as well as $(0)$ again, a non-closed point that is thought of as "somewhere in $\mathbb{C}^2$, but nowhere in particular". It also consists of non-closed points corresponding to irreducible polynomials $f(x, y)$, for example $(y - x^2)$, which is a "generic point on the parabola".

(c) If $k$ is a field, $\operatorname{Spec} k$ is a single point, since the only maximal ideal of $k$ is $(0)$.

**Example 84.2.4** (Complex affine varieties)

Let $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ be an ideal. By Proposition 83.6.1, the set

$$\operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]/I$$

consists of those prime ideals of $\mathbb{C}[x_1, \dots, x_n]$ which contain $I$: in other words, it has a point for every closed irreducible subvariety of $\mathcal{V}(I)$. So in addition to the "geometric points" (corresponding to the maximal ideals $(x_1 - a_1, \dots, x_n - a_n)$ we have non-closed points along each of the varieties).

The non-closed points are the ones you are not used to: there is one for each non-maximal prime ideal (visualized as "irreducible subvariety"). I like to visualize them in my head like a fly: you can hear it, so you know it is floating *somewhere* in the room, but as it always moving, you never know exactly where. So the generic point of $\operatorname{Spec} \mathbb{C}[x, y]$ corresponding to the prime ideal $(0)$ is floating everywhere in the plane, the one for the ideal $(y - x^2)$ floats along the parabola, etc.

Image from [**Wa**].

**Remark 84.2.5** (Why don't the prime non-maximal ideals correspond to the whole parabola?) — We have already seen a geometric reason in Section 83.4 earlier: localizing a ring at a prime non-maximal ideal gives the functions that may blow up somewhere in the parabola, but not *generically*.

**Example 84.2.6** (More examples of spectrums)

(a) $\operatorname{Spec} \mathbb{Z}$ consists of a point for every prime $p$, plus a generic point that is somewhere, but nowhere in particular.

(b) $\operatorname{Spec} \mathbb{C}[x]/(x^2)$ has only $(x)$ as a prime ideal. The ideal $(0)$ is not prime since $0 = x \cdot x$. Thus as a *topological space*, $\operatorname{Spec} \mathbb{C}[x]/(x^2)$ is a single point.

(c) $\operatorname{Spec} \mathbb{Z}/60\mathbb{Z}$ consists of three points. What are they?

## §84.3 The Zariski topology on the spectrum

*Prototypical example for this section: Still* $\operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]/I$.

Now, we endow a topology on $\operatorname{Spec} A$. Since the points on $\operatorname{Spec} A$ are the prime ideals, we continue the analogy by thinking of the points $f$ as functions on $\operatorname{Spec} A$. That is:

**Definition 84.3.1.** Let $f \in A$ and $\mathfrak{p} \in \operatorname{Spec} A$. Then the **value** of $f$ at $\mathfrak{p}$ is defined to be $f \pmod{\mathfrak{p}}$, an element of $A/\mathfrak{p}$. We denote it $f(\mathfrak{p})$.

**Example 84.3.2** (Vanishing locii in $\mathbb{A}^n$)

Suppose $A = \mathbb{C}[x_1, \dots, x_n]$, and $\mathfrak{m} = (x_1 - a_1, x_2 - a_2, \dots, x_n - a_n)$ is a maximal ideal of $A$. Then for a polynomial $f \in \mathbb{C}$,

$$f \pmod{\mathfrak{m}} = f(a_1, \dots, a_n)$$

with the identification that $A/\mathfrak{m} \cong \mathbb{C}$.

**Example 84.3.3** (Functions on $\operatorname{Spec} \mathbb{Z}$)

Consider $A = \operatorname{Spec} \mathbb{Z}$. Then 2019 is a function on $A$. Its value at the point $(5)$ is $4 \pmod 5$; its value at the point $(7)$ is $3 \pmod 7$.

Indeed if you replace $A$ with $\mathbb{C}[x_1, \dots, x_n]$ and $\operatorname{Spec} A$ with $\mathbb{A}^n$ in everything that follows, then everything will become quite familiar.

**Definition 84.3.4.** Let $f \in A$. We define the **vanishing locus** of $f$ to be

$$\mathcal{V}(f) = \{\mathfrak{p} \in \operatorname{Spec} A \mid f(\mathfrak{p}) = 0\} = \{\mathfrak{p} \in \operatorname{Spec} A \mid f \in \mathfrak{p}\}.$$

More generally, just as in the affine case, we define the vanishing locus for an ideal $I$ as

$$\begin{aligned}\mathcal{V}(I) &= \{\mathfrak{p} \in \operatorname{Spec} A \mid f(\mathfrak{p}) = 0 \, \forall f \in I\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} A \mid f \in \mathfrak{p} \, \forall f \in I\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} A \mid I \subseteq \mathfrak{p}\}.\end{aligned}$$

Finally, we define the **Zariski topology** on $\operatorname{Spec} A$ by declaring that the sets of the form $\mathcal{V}(I)$ are closed.

We now define a few useful topological notions:

**Definition 84.3.5.** Let $X$ be a topological space. A point $p \in X$ is a **closed point** if the set $\{p\}$ is closed.

> **Question 84.3.6** (Mandatory)**.** Show that a point (i.e. prime ideal) $\mathfrak{m} \in \operatorname{Spec} A$ is a closed point if and only if $\mathfrak{m}$ is a maximal ideal.

Recall also in Definition 7.2.4 we denote by $\overline{S}$ the closure of a set $S$ (i.e. the smallest closed set containing $S$); so you can think of a closed point $p$ also as one whose closure is just $\{p\}$. Therefore the Zariski topology lets us refer back to the old "geometric" as just the closed points.

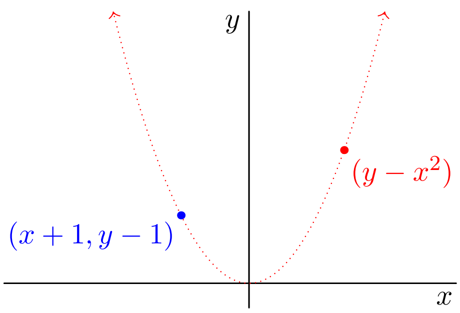
**Example 84.3.7** (Non-closed points, continued)

Let $A = \mathbb{C}[x, y]$ and let $\mathfrak{p} = (y - x^2) \in \operatorname{Spec} A$; this is the "generic point" on a parabola. It is not closed, but we can compute its closure:

$$\overline{\{\mathfrak{p}\}} = \mathcal{V}(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} A \mid \mathfrak{q} \supseteq \mathfrak{p}\}.$$

This closure contains the point $\mathfrak{p}$ as well as several maximal ideals $\mathfrak{q}$, such as $(x - 2, y - 4)$ and $(x - 3, y - 9)$. In other words, the closure of the "generic point" of the parabola is literally the set of all points that are actually on the parabola (including generic points).

That means the way to picture $\mathfrak{p}$ is a point that is floating "somewhere on the parabola", but nowhere in particular. It makes sense then that if we take the closure, we get the entire parabola, since $\mathfrak{p}$ "could have been" any of those points.



**Example 84.3.8** (The generic point of the $y$-axis isn't on the $x$-axis)
Let $A = \mathbb{C}[x, y]$ again. Consider $\mathcal{V}(y)$, which is the $x$-axis of $\operatorname{Spec} A$. Then consider $\mathfrak{p} = (x)$, which is the generic point on the $y$-axis. Observe that

$$\mathfrak{p} \notin \mathcal{V}(y).$$

The geometric way of saying this is that a *generic point* on the $y$-axis does not lie on the $x$-axis.

We now also introduce one more word:

**Definition 84.3.9.** A topological space $X$ is **irreducible** if either of the following two conditions hold:

- The space $X$ cannot be written as the union of two proper closed subsets.
- Any two nonempty open sets of $X$ intersect.

A subset $Z$ of $X$ (usually closed) is irreducible if it is irreducible as a subspace.

**Exercise 84.3.10.** Show that the two conditions above are indeed equivalent. Also, show that the closure of a point is always irreducible.

This is the analog of the "irreducible" we defined for affine varieties, but it is now a topological definition, although in practice this definition is only useful for spaces with the Zariski topology. Indeed, if any two nonempty open sets intersect (and there is more than one point), the space is certainly not Hausdorff! As with our old affine varieties, the intuition is that $\mathcal{V}(xy)$ (the union of two lines) should not be irreducible.

**Example 84.3.11** (Reducible and irreducible spaces)

(a) The closed set $\mathcal{V}(xy) = \mathcal{V}(x) \cup \mathcal{V}(y)$ is reducible.

(b) The entire plane $\operatorname{Spec} \mathbb{C}[x, y]$ is irreducible. There is actually a simple (but counter-intuitive, since you are just getting used to generic points) reason why this is true: the generic point $(0)$ is in *every* open set, ergo, any two open sets intersect.

So actually, the generic points kind of let us cheat our way through the following bit:

**Proposition 84.3.12** (Spectrums of integral domains are irreducible)
If $A$ is an integral domain, then $\operatorname{Spec} A$ is irreducible.

*Proof.* Just note $(0)$ is a prime ideal, and is in every open set. $\square$

You should compare this with our old classical result that $\mathbb{C}[x_1, \dots, x_n]/I$ was irreducible as an affine variety exactly when $I$ was prime. This time, the generic point actually takes care of the work for us: the fact that it is *allowed* to float anywhere in the plane lets us capture the idea that $\mathbb{A}^2$ should be irreducible without having to expend any additional effort.

**Remark 84.3.13** — Surprisingly, the converse of this proposition is false: we have seen $\operatorname{Spec}\mathbb{C}[x]/(x^2)$ has only one point, so is certainly irreducible. But $A = \mathbb{C}[x]/(x^2)$ is not an integral domain. So this is one weird-ness introduced by allowing "non-radical" behavior.

At this point you might notice something:

**Theorem 84.3.14** (Points are in bijection with irreducible closed sets)
Consider $X = \operatorname{Spec} A$. For every irreducible closed set $Z$, there is exactly one point $\mathfrak{p}$ such that $Z = \overline{\{\mathfrak{p}\}}$. (In particular points of $X$ are in bijection with closed subsets of $X$.)

*Idea of proof.* The point $\mathfrak{p}$ corresponds to the closed set $\mathcal{V}(\mathfrak{p})$, which one can show is irreducible. □

This gives you a better way to draw non-closed points: they are the generic points lying along any irreducible closed set (consisting of more than just one point).
At this point,[1] I may as well give you the real definition of generic point.

**Definition 84.3.15.** Given a topological space $X$, a **generic point** $\eta$ is a point whose closure is the entire space $X$.

So for us, when $A$ is an integral domain, $\operatorname{Spec} A$ has generic point $(0)$.

**Abuse of Notation 84.3.16.** Very careful readers might note I am being a little careless with referring to $(y-x^2)$ as "the generic point along the parabola" in $\operatorname{Spec}\mathbb{C}[x, y]$. What's happening is that $\mathcal{V}(y-x^2)$ is a closed set, and as a topological subspace, it has generic point $(y-x^2)$.

## §84.4 Krull dimension

Surprisingly, the topology is good enough to give dimension. For affine schemes:

**Definition 84.4.1.** Let $A$ be a commutative ring. Consider chains of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subseteq A$, where $n$ is called the *length*. The supremum of all possible $n$ is the called **Krull dimension** of $A$.
The Krull dimension is always nonnegative unless $A$ is the zero ring, in which case either $-1$ or $-\infty$ are used conventionally.

This definition should match your intuition.

**Example 84.4.2** (Examples of Krull dimension)
(a) $\mathbb{C}[x_1, \dots, x_n]$ has Krull dimension $n$, with the chain

$$(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, \dots, x_n)$$

having length $n$. This matches our expectation that $\operatorname{Spec}\mathbb{C}[x_1, \dots, x_n]$ corresponds to $\mathbb{A}^n$.

[1] Pun not intended

(b) $\mathbb{C}[x,y]/(y-x^2)$ has Krull dimension 1, with the chain

$$(x,y) \subsetneq (y-x^2)$$

having length 1. Geometrically, we think of $(x,y)$ as the origin and $(y-x^2)$ as the parabola itself.

(c) $\mathbb{Z}$ has Krull dimension 1.

(d) $\mathbb{Z}/(60)$ has Krull dimension 0; it's just three points.

You can do this more generally with a topological space $X$: the Krull dimension of a space is the supremum of chains of irreducible closed subspaces $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$. You'd only want to use this definition in situations where $X$ had a Zariski topology: in particular, if $X = \operatorname{Spec} A$, this is just the Krull dimension of the ring $A$ itself.

## §84.5 On radicals

Back when we studied classical algebraic geometry in $\mathbb{C}^n$, we saw Hilbert's Nullstellensatz (Theorem 77.3.4) show up to give bijections between radical ideals and affine varieties; we omitted the proof, because it was nontrivial.

However, for a *scheme*, where the points *are* prime ideals (rather than tuples in $\mathbb{C}^n$), the corresponding results will actually be *easy*: even in the case where $A = \mathbb{C}[x_1, \dots, x_n]$, the addition of prime ideals (instead of just maximal ideals) will actually *simplify* the proof, because radicals play well with prime ideals.

We still have the following result.

**Proposition 84.5.1** ($\mathcal{V}(\sqrt{I}) = \mathcal{V}(I)$)
For any ideal $I$ of a ring $A$ we have $\mathcal{V}(\sqrt{I}) = \mathcal{V}(I)$.

*Proof.* We have $\sqrt{I} \supseteq I$. Hence automatically $\mathcal{V}(\sqrt{I}) \subseteq \mathcal{V}(I)$.

Conversely, if $\mathfrak{p} \in \mathcal{V}(I)$, then $I \subseteq \mathfrak{p}$, so $\sqrt{I} \subseteq \sqrt{\mathfrak{p}} = \mathfrak{p}$ (by Proposition 77.3.3). $\square$

We hinted the key result in an earlier remark, and we now prove it.

**Theorem 84.5.2** (Radical is intersection of primes)
Let $I$ be an ideal of a ring $A$. Then

$$\sqrt{I} = \bigcap_{\mathfrak{p} \supseteq I} \mathfrak{p}.$$

*Proof.* This is a famous statement from commutative algebra, and we prove it here only for completeness. It is "doing most of the work".

Note that if $I \subseteq \mathfrak{p}$, then $\sqrt{I} \subseteq \sqrt{\mathfrak{p}} = \mathfrak{p}$; thus $\sqrt{I} \subseteq \bigcap_{\mathfrak{p} \supseteq I} \mathfrak{p}$.

Conversely, suppose $x \notin \sqrt{I}$, meaning $1, x, x^2, x^3, \dots \notin I$. Then, consider the localization $(A/I)[1/x]$, which is not the zero ring. Like any ring, it has some maximal ideal (Krull's theorem). This means our usual bijection between prime ideals of $(A/I)[1/x]$, prime ideals of $A/I$ and prime ideals of $A$ gives some prime ideal $\mathfrak{p}$ of $A$ containing $I$ but not containing $x$. Thus $x \notin \bigcap_{\mathfrak{p} \supseteq I} \mathfrak{p}$, as desired.

The key idea here is, for $x \in A$, $x^n = 0$ for some positive *finite* integer $n$ if and only if $A[1/x] = 0$.

So, in other words,

$$\begin{aligned} & x \in \sqrt{(0)} \\ & \iff x^n = 0 \text{ for some positive integer } n \\ & \iff A[1/x] = 0 \\ & \iff \text{for all prime ideals } \mathfrak{p},\ x \in \mathfrak{p} \\ & \iff x \in \bigcap_{\mathfrak{p}} \mathfrak{p}. \end{aligned}$$

When $I$ is not $(0)$, consider the ring $A/I$ instead. $\square$

Geometrically speaking, this theorem states:

For any $f$ a regular function on $\operatorname{Spec} A/I$, then

$$f^n = 0 \text{ for some positive integer } n \iff f \text{ vanishes at all points in } \operatorname{Spec} A/I.$$

To which, the proof above reads:

$$\begin{aligned} & f \in \sqrt{(I)} \\ & \iff f^n \in I \text{ for some positive integer } n \\ & \iff (A/I)[1/f] = 0 \\ & \iff \operatorname{Spec}(A/I)[1/f] \text{ is empty} \\ & \iff \text{for all } \mathfrak{p} \in \operatorname{Spec} A/I,\ f \text{ vanishes at } \mathfrak{p} \\ & \iff f \in \bigcap_{\mathfrak{p} \in \operatorname{Spec} A/I} \mathfrak{p}. \end{aligned}$$

You may want to run through the proof with the example $A = k[x]$, $I = (x^2)$ and $f = x$ in Section 86.7, keeping in mind the image of $\operatorname{Spec} A/I$ as a "fuzzy" point and $f$ being a nonzero function that takes value zero at every point.

**Remark 84.5.3** (A variant of Krull's theorem) — The longer direction of this proof is essentially saying that for any $x \in A$, there is a maximal ideal of $A$ not containing $x$. The "short" proof is to use Krull's theorem on $(A/I)[1/x]$ as above, but one can also still prove it directly using Zorn's lemma (by copying the proof of the original Krull's theorem).

**Example 84.5.4** ($\sqrt{(2016)} = (42)$ in $\mathbb{Z}$)

In the ring $\mathbb{Z}$, we see that $\sqrt{(2016)} = (42)$, since the distinct primes containing $(2016)$ are $(2)$, $(3)$, $(7)$.

Geometrically, this gives us a good way to describe $\sqrt{I}$: it is the *set of all functions vanishing on all of* $\mathcal{V}(I)$. Indeed, we may write

$$\sqrt{I} = \bigcap_{\mathfrak{p} \supseteq I} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \mathcal{V}(I)} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \mathcal{V}(I)} \{ f \in A \mid f(\mathfrak{p}) = 0 \}.$$

We can now state:

**Theorem 84.5.5** (Radical ideals correspond to closed sets)
Let $I$ and $J$ be ideals of $A$, and considering the space $\operatorname{Spec} A$. Then

$$\mathcal{V}(I) = \mathcal{V}(J) \iff \sqrt{I} = \sqrt{J}.$$

In particular, radical ideals exactly correspond to closed subsets of $\operatorname{Spec} A$.

*Proof.* If $\mathcal{V}(I) = \mathcal{V}(J)$, then $\sqrt{I} = \bigcap_{\mathfrak{p}\in\mathcal{V}(I)} \mathfrak{p} = \bigcap_{\mathfrak{p}\in\mathcal{V}(J)} \mathfrak{p} = \sqrt{J}$ as needed.
Conversely, suppose $\sqrt{I} = \sqrt{J}$. Then $\mathcal{V}(I) = \mathcal{V}(\sqrt{I}) = \mathcal{V}(\sqrt{J}) = \mathcal{V}(J)$. $\square$

Compare this to the theorem we had earlier that the *irreducible* closed subsets correspond to *prime* ideals!

## §84.6 A few harder problems to think about

As Chapter 86 contains many examples of affine schemes to train your intuition, it's possibly worth reading even before attempting these problems, even though there will be some parts that won't make sense yet.

**Problem 84A** (Spec $\mathbb{Q}[x]$)**.** Describe the points and topology of $\operatorname{Spec} \mathbb{Q}[x]$.

**Problem 84B** (Product rings)**.** Describe the points and topology of $\operatorname{Spec} A \times B$ in terms of $\operatorname{Spec} A$ and $\operatorname{Spec} B$.

**Problem 84C**$^\star$ (How to actually think about Artinian rings)**.** Let $A$ be a Noetherian ring. Prove that all of the following are equivalent:

(i) $A$ is Artinian, i.e. satisfies the descending chain condition (as defined in Problem 4H).

(ii) $A$ has Krull dimension 0 or $A$ is the zero ring.

(iii) $\operatorname{Spec} A$ is finite and discrete.

(iv) $\operatorname{Spec} A$ is finite.

# 85 Affine schemes: the sheaf

We now complete our definition of $X = \operatorname{Spec} A$ by defining the sheaf $\mathcal{O}_X$ on it, making it into a ringed space. This is done quickly in the first section.

As before, our goal is:

**The sheaf $\mathcal{O}_X$ coincides with the sheaf of regular functions on affine varieties, so that we can apply our geometric intuition to $\operatorname{Spec} A$ when $A$ is an arbitrary ring.**

However, we will then spend the next several chapters trying to convince the reader to *forget* the definition we gave, in practice. This is because practically, the sections of the sheaves are best computed by not using the definition directly, but by using some other results.

Along the way we'll develop some related theory: in computing the stalks we'll find out the definition of a local ring, and in computing the sections we'll find out about distinguished open sets.

A reminder once again: Chapter 86 has many more concrete examples. It's not a bad idea to look through there for more examples if anything in this chapter trips you up.

## §85.1 A useless definition of the structure sheaf

*Prototypical example for this section: Still* $\mathbb{C}[x_1, \dots, x_n]/I$.

We have now endowed $\operatorname{Spec} A$ with the Zariski topology, and so all that remains is to put a sheaf $\mathcal{O}_{\operatorname{Spec} A}$ on it. To do this we want a notion of "regular functions" as before.

This is easy to do since we have localizations on hand.

**Definition 85.1.1.** First, let $\mathscr{F}$ be the pre-sheaf of "globally rational" functions: i.e. we define $\mathscr{F}(U)$ to be the localization

$$\mathscr{F}(U) = \left\{ \frac{f}{g} \mid f, g \in A \text{ and } g(\mathfrak{p}) \neq 0 \; \forall \mathfrak{p} \in U \right\} = \left( A \setminus \bigcup_{\mathfrak{p} \in U} \mathfrak{p} \right)^{-1} A.$$

We now define the structure sheaf on $\operatorname{Spec} A$. It is

$$\mathcal{O}_{\operatorname{Spec} A} = \mathscr{F}^{\text{sh}}$$

i.e. the sheafification of the $\mathscr{F}$ we just defined.

**Exercise 85.1.2.** Compare this with the definition for $\mathcal{O}_V$ with $V$ a complex variety, and check that they essentially match.

And thus, we have completed the transition to adulthood, with a complete definition of the affine scheme.

If you really like compatible germs, you can write out the definition:

**Definition 85.1.3.** Let $A$ be a ring. Then $\operatorname{Spec} A$ is made into a ringed space by setting

$$\mathcal{O}_{\operatorname{Spec} A}(U) = \{(f_\mathfrak{p} \in A_\mathfrak{p})_{\mathfrak{p}\in U} \text{ which are locally quotients}\}.$$

That is, it consists of sequence $(f_\mathfrak{p})_{\mathfrak{p}\in U}$, with each $f_\mathfrak{p} \in A_\mathfrak{p}$, such that for every point $\mathfrak{p}$ there is an open neighborhood $U_\mathfrak{p}$ and an $f, g \in A$ such that $f_\mathfrak{q} = \frac{f}{g} \in A_\mathfrak{q}$ for all $\mathfrak{q} \in U_\mathfrak{p}$.

We will now **basically forget about this definition**, because we will never use it in practice. In the next two sections, we will show you:

- that the stalks $\mathcal{O}_{\operatorname{Spec} A,\mathfrak{p}}$ are just $A_\mathfrak{p}$, and
- that the sections $\mathcal{O}_{\operatorname{Spec} A}(U)$ can be computed, for any open set $U$, by focusing only on the special case where $U = D(f)$ is a distinguished open set.

These two results will be good enough for all of our purposes, so we will be able to not use this definition. (Hence the lack of examples in this section.)

## §85.2 The value of distinguished open sets (or: how to actually compute sections)

*Prototypical example for this section: $D(x)$ in $\operatorname{Spec} \mathbb{C}[x]$ is the punctured line.*

We will now really hammer in the importance of the distinguished open sets. The definition is analogous to before:

**Definition 85.2.1.** Let $f \in \operatorname{Spec} A$. Then $D(f)$ is the set of $\mathfrak{p}$ such that $f(\mathfrak{p}) \neq 0$, a **distinguished open set**.

Distinguished open sets will have three absolutely crucial properties, which build on each other.

### §85.2.i A basis of the Zariski topology

The first is a topological observation:

**Theorem 85.2.2** (Distinguished open sets form a base)
The distinguished open sets $D(f)$ form a basis for the Zariski topology: any open set $U$ is a union of distinguished open sets.

*Proof.* Let $U$ be an open set; suppose it is the complement of closed set $V(I)$. Then verify that

$$U = \bigcup_{f\in I} D(f).$$ □

### §85.2.ii Sections are computable

The second critical fact is that the sections on distinguished open sets can be computed explicitly.

**Theorem 85.2.3** (Sections of $D(f)$ are localizations away from $f$)
Let $A$ be a ring and $f \in A$. Then

$$\mathcal{O}_{\operatorname{Spec} A}(D(f)) \cong A[1/f].$$

*Proof.* Omitted, but similar to Theorem 78.6.1. □

**Example 85.2.4** (The punctured line is isomorphic to a hyperbola)
The "hyperbola effect" appears again:

$$\mathcal{O}_{\operatorname{Spec}\mathbb{C}[x]}(D(x)) = \mathbb{C}[x, x^{-1}] \cong \mathbb{C}[x, y]/(xy - 1).$$

On a tangential note, we had better also note somewhere that $\operatorname{Spec} A = D(1)$ is itself distinguished open, so the global sections can be recovered.

**Corollary 85.2.5** ($A$ is the ring of global sections)
The ring of global sections of $\operatorname{Spec} A$ is $A$.

*Proof.* By previous theorem, $\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = \mathcal{O}_{\operatorname{Spec} A}(D(1)) = A[1/1] = A$. □

## §85.2.iii They are affine

We know $\mathcal{O}_X(D(f)) = A[1/f]$. In fact, if you draw $\operatorname{Spec} A[1/f]$, you will find that it looks exactly like $D(f)$. So the third final important fact is that $D(f)$ will actually be *isomorphic* to $\operatorname{Spec} A[1/f]$ (just like the line minus the origin is isomorphic to the hyperbola). We can't make this precise yet, because we have not yet discussed morphisms of schemes, but it will be handy later (though not right away).

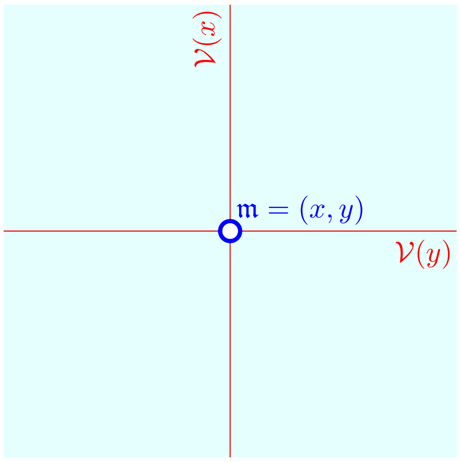
## §85.2.iv Classic example: the punctured plane

We now give the classical example of a computation which shows how you can forget about sheafification, if you never liked it.[1] The idea is that:

**We can compute any section $\mathcal{O}_X(U)$ in practice by using distinguished open sets and sheaf axioms.**

Let $X = \operatorname{Spec}\mathbb{C}[x, y]$, and consider the origin, i.e. the point $\mathfrak{m} = (x, y)$. This ideal is maximal, so it corresponds to a closed point, and we can consider the open set $U$ consisting of all the points other than $\mathfrak{m}$. We wish to compute $\mathcal{O}_X(U)$.

[1]This perspective is so useful that some sources, like Vakil [Va17, §4.1] will *define* $\mathcal{O}_{\operatorname{Spec} A}$ by requiring $\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A[1/f]$, rather than use sheafification as we did.




Unfortunately, $U$ is not distinguished open. But, we can compute it anyways by writing $U = D(x) \cup D(y)$: conveniently, $D(x) \cap D(y) = D(xy)$. By the sheaf axioms, we have a pullback square

$$\begin{tikzcd}
\mathcal{O}_X(U) \arrow[r] \arrow[d] & \mathcal{O}_X(D(x)) = \mathbb{C}[x, y, x^{-1}] \arrow[d] \\
\mathcal{O}_X(D(y)) = \mathbb{C}[x, y, y^{-1}] \arrow[r] & \mathcal{O}_X(D(xy)) = \mathbb{C}[x, y, x^{-1}, y^{-1}].
\end{tikzcd}$$

In other words, $\mathcal{O}_X(U)$ consists of pairs

$$f \in \mathbb{C}[x, y, x^{-1}]$$
$$g \in \mathbb{C}[x, y, y^{-1}]$$

which agree on the overlap: $f = g$ on $D(x) \cap D(y)$. Well, we can describe $f$ as a polynomial with some $x$'s in the denominator, and $g$ as a polynomial with some $y$'s in the denominator. If they match, the denominator is actually constant. Put crudely,

$$\mathbb{C}[x, y, x^{-1}] \cap \mathbb{C}[x, y, y^{-1}] = \mathbb{C}[x, y].$$

In conclusion,

$$\mathcal{O}_X(U) = \mathbb{C}[x, y].$$

That is, we get no additional functions.

## §85.3 The stalks of the structure sheaf

*Prototypical example for this section: The stalk of* $\operatorname{Spec} \mathbb{C}[x, y]$ *at* $\mathfrak{m} = (x, y)$ *are rational functions defined at the origin.*

Don't worry, this one is easier than last section.

### §85.3.i They are localizations

**Theorem 85.3.1** (Stalks of $\operatorname{Spec} A$ are $A_\mathfrak{p}$)
Let $A$ be a ring and let $\mathfrak{p} \in \operatorname{Spec} A$. Then

$$\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} \cong A_\mathfrak{p}.$$

In particular $\operatorname{Spec} A$ is a locally ringed space.

*Proof.* Since sheafification preserved stalks, it's enough to check it for $\mathscr{F}$ the pre-sheaf of globally rational functions in our definition. The proof is basically the same as Theorem 82.3.7: there is an obvious map $\mathscr{F}_\mathfrak{p} \to A_\mathfrak{p}$ on germs by

$$(U, f/g \in \mathscr{F}(U)) \mapsto f/g \in A_\mathfrak{p}.$$

(Note the $f/g$ on the left lives in $\mathscr{F}(U)$ but the one on the right lives in $A_\mathfrak{p}$). We show injectivity and surjectivity:

- Injective: suppose $(U_1, f_1/g_1)$ and $(U_2, f_2/g_2)$ are two germs with $f_1/g_1 = f_2/g_2 \in A_\mathfrak{p}$. This means $h(g_1 f_2 - f_2 g_1) = 0$ in $A$, for some nonzero $h$. Then both germs identify with the germ $(U_1 \cap U_2 \cap D(h), f_1/g_1)$.
- Surjective: let $U = D(g)$. □

**Example 85.3.2** (Denominators not divisible by $x$)

We have seen this example so many times that I will only write it in the new notation, and make no further comment: if $X = \operatorname{Spec} \mathbb{C}[x]$ then

$$\mathscr{O}_{\operatorname{Spec} X,(x)} = \mathbb{C}[x]_{(x)} = \left\{ \frac{f}{g} \mid g(0) \neq 0 \right\}.$$

**Example 85.3.3** (Denominators not divisible by $x$ or $y$)

Let $X = \operatorname{Spec} \mathbb{C}[x,y]$ and let $\mathfrak{m} = (x,y)$ be the origin. Then

$$\mathbb{C}[x,y]_{(x,y)} = \left\{ \frac{f(x,y)}{g(x,y)} \mid g(0,0) \neq 0 \right\}.$$

If you want more examples, take any of the ones from Section 83.4, and try to think about what they mean geometrically.

## §85.3.ii Motivating local rings: germs should package values

Let's return to our well-worn example $X = \operatorname{Spec} \mathbb{C}[x,y]$ and consider $\mathfrak{m} = (x,y)$ the origin. The stalk was

$$\mathscr{O}_{X,\mathfrak{m}} = \mathbb{C}[x,y]_{(x,y)} = \left\{ \frac{f(x,y)}{g(x,y)} \mid g(0,0) \neq 0 \right\}.$$

So let's take some section like $f = \frac{1}{xy+4}$, which is a section of $U = D(xy+4)$ (or some smaller open set, but we'll just use this one for simplicity). We also have $U \ni m$, and so $f$ gives a germ at $\mathfrak{m}$.

On the other hand, $f$ also has a value at $\mathfrak{m}$: it is $f \pmod{\mathfrak{m}} = \frac{1}{4}$. And in general, the ring of possible values of a section at the origin $\mathfrak{m}$ is $\mathbb{C}[x,y]/\mathfrak{m} \cong \mathbb{C}$.

Now, you might recall that I pressed the point of view that a germ might be thought of as an "enriched value". Then it makes sense that if you know the germ of a section $f$ at a point $\mathfrak{m}$ — i.e., you know the "enriched value" — then you should be able to compute its value as well. What this means is that we ought to have some map

$$A_\mathfrak{m} \to A/\mathfrak{m}$$

sending germs to their associated values.

Indeed you can, and this leads us to...

## §85.4 Local rings and residue fields: linking germs to values

*Prototypical example for this section: The residue field of* $\operatorname{Spec} \mathbb{C}[x, y]$ *at* $\mathfrak{m} = (x, y)$ *is* $\mathbb{C}$.

### §85.4.i Localizations give local rings

This notation is about to get really terrible, but bear with me.

**Theorem 85.4.1** (Stalks are local rings)

Let $A$ be a ring and $\mathfrak{p}$ any prime ideal. Then the localization $A_\mathfrak{p}$ has exactly one maximal ideal, given explicitly by

$$\mathfrak{p}A_\mathfrak{p} = \left\{ \frac{f}{g} \mid f \in \mathfrak{p},\ g \notin \mathfrak{p} \right\}.$$

The ideal $\mathfrak{p}A_\mathfrak{p}$ thus captures the idea of "germs vanishing at $\mathfrak{p}$".[2]

Proof in a moment; for now let's introduce some words so we can give our examples in the proper language.

**Definition 85.4.2.** A ring $R$ with exactly one maximal ideal $\mathfrak{m}$ will be called a **local ring**. The **residue field** is the quotient $A/\mathfrak{m}$.

**Question 85.4.3.** Are fields local rings?

Thus what we find is that:

**The stalks consist of the possible enriched values (germs); the residue field is the set of (un-enriched) values.**

**Example 85.4.4** (The stalk at the origin of $\operatorname{Spec} \mathbb{C}[x, y]$)

Again set $A = \mathbb{C}[x, y]$, $X = \operatorname{Spec} A$ and $\mathfrak{p} = (x, y)$ so that $\mathcal{O}_{X,\mathfrak{p}} = A_\mathfrak{p}$. (I switched to $\mathfrak{p}$ for the origin, to avoid confusion with the maximal ideal $\mathfrak{p}A_\mathfrak{p}$ of the local ring $A_\mathfrak{p}$.) As we said many times already, $A_\mathfrak{p}$ consists of rational functions not vanishing at the origin, such as $f = \frac{1}{xy+4}$.

What is the unique maximal ideal $\mathfrak{p}A_\mathfrak{p}$? Answer: it consists of the rational functions which *vanish* at the origin: for example, $\frac{x}{x^2+3y}$, or $\frac{3x+5y}{2}$, or $\frac{-xy}{4(xy+4)}$. If we allow ourselves to mod out by such functions, we get the residue field $\mathbb{C}$, and $f$ will have the value $\frac{1}{4}$, since

$$\frac{1}{xy+4} - \underbrace{\frac{-xy}{4(xy+4)}}_{\text{vanishes at origin}} = \frac{1}{4}.$$

More generally, suppose $f$ is any section of some open set containing $\mathfrak{p}$. Let $c \in \mathbb{C}$ be the value $f(\mathfrak{p})$, that is, $f \pmod{\mathfrak{p}}$. Then $f - c$ is going to be another section which vanishes at the origin $\mathfrak{p}$, so as promised, $f \equiv c \pmod{\mathfrak{p}A_\mathfrak{p}}$.

[2] The notation $\mathfrak{p}A_\mathfrak{p}$ really means the set of $f \cdot h$ where $f \in \mathfrak{p}$ (viewed as a subset of $A_\mathfrak{p}$ by $f \mapsto \frac{f}{1}$) and $h \in A_\mathfrak{p}$. I personally find this is more confusing than helpful, so I'm footnoting it.

Okay, we can write down a proof of the theorem now.

*Proof of Theorem 85.4.1.* One may check that the set $I = \mathfrak{p}A_\mathfrak{p}$ is an ideal of $A_\mathfrak{p}$. Moreover, $1 \notin I$, so $I$ is proper.

To prove it is maximal and unique, it suffices to prove that any $f \in A_\mathfrak{p}$ with $f \notin I$ is a *unit* of $A_\mathfrak{p}$. This will imply $I$ is maximal: there are no more non-units to add. It will also imply $I$ is the only maximal ideal: because any proper ideal can't contain units, so is contained in $I$.

This is actually easy. An element of $A_\mathfrak{p}$ not in $I$ must be $x = \frac{f}{g}$ for $f, g \in A$ and $f, g \notin \mathfrak{p}$. For such an element, $x^{-1} = \frac{g}{f} \notin \mathfrak{p}$ too. So $x$ is a unit. End proof. $\square$

Even more generally:

**If a sheaf $\mathscr{F}$ consists of "field-valued functions", the stalk $\mathscr{F}_p$ probably has a maximal ideal consisting of the germs vanishing at $p$.**

**Example 85.4.5** (Local rings in non-algebraic geometry sheaves)

Let's go back to the example of $X = \mathbb{R}$ and $\mathscr{F}(U)$ the smooth functions, and consider the stalk $\mathscr{F}_p$, where $p \in X$. Define the ideal $\mathfrak{m}_p$ to be the set of germs $(s, U)$ for which $s(p) = 0$.

Then $\mathfrak{m}_p$ is maximal: we have an exact sequence

$$0 \to \mathfrak{m}_p \to \mathscr{F}_p \xrightarrow{(s,U)\mapsto s(p)} \mathbb{R} \to 0$$

and so $\mathscr{F}_p/\mathfrak{m}_p \cong \mathbb{R}$, which is a field.

It remains to check there are no nonzero maximal ideals. Now note that if $s \notin \mathfrak{m}_p$, then $s$ is nonzero in some open neighborhood of $p$, and one can construct the function $1/s$ on it. So **every element of $\mathscr{F}_p \setminus \mathfrak{m}_p$ is a unit**; and again $\mathfrak{m}_p$ is in fact the only maximal ideal!

Thus the stalks of each of the following types of sheaves are local rings, too.

- Sheaves of continuous real/complex functions on a topological space
- Sheaves of smooth functions on any manifold
- etc.

## §85.4.ii Computing values: a convenient square

Very careful readers might have noticed something a little uncomfortable in our extended example with $\operatorname{Spec} A$ with $A = \mathbb{C}[x, y]$ and $\mathfrak{p} = (x, y)$ the origin. Let's consider $f = \frac{1}{xy+4}$. We took $f \pmod{x, y}$ in the original ring $A$ in order to decide the value "should" be $\frac{1}{4}$. However, all our calculations actually took place not in the ring $A$, but instead in the ring $A_\mathfrak{p}$. Does this cause issues?

Thankfully, no, nothing goes wrong, even in a general ring $A$.

**Definition 85.4.6.** We let the quotient $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, i.e. the **residue field** of the stalk of $\operatorname{Spec} A$ at $\mathfrak{p}$, be denoted by $\kappa(\mathfrak{p})$.

Then the following is a special case of Theorem 83.7.1 (localization commutes with quotients):

**Theorem 85.4.7** (The germ-to-value square)

Let $A$ be a ring and $\mathfrak{p}$ a prime ideal. The following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\text{localize}"] \arrow[d, "\bmod \mathfrak{p}"'] & A_\mathfrak{p} \arrow[d, "\bmod \mathfrak{p}"] \\ A/\mathfrak{p} \arrow[r, "\operatorname{Frac}(-)"] & \kappa(\mathfrak{p}) \end{tikzcd}$$

In particular, $\kappa(\mathfrak{p})$ can also be described as $\operatorname{Frac}(A/\mathfrak{p})$.

So for example, if $A = \mathbb{C}[x,y]$ and $\mathfrak{p} = (x,y)$, then $A/\mathfrak{p} = \mathbb{C}$ and $\operatorname{Frac}(A/\mathfrak{p}) = \operatorname{Frac}(\mathbb{C}) = \mathbb{C}$, as we expected. In practice, $\operatorname{Frac}(A/\mathfrak{p})$ is probably the easier way to compute $\kappa(\mathfrak{p})$ for any prime ideal $\mathfrak{p}$.

## §85.5 Recap

To recap the last two chapters, let $A$ be a ring.

- We define $X = \operatorname{Spec} A$ to be the set of prime ideals of $A$.
  - The maximal ideals are the "closed points" we are used to, but the prime ideals are "generic points".
- We equip $\operatorname{Spec} A$ with the Zariski topology by declaring $\mathcal{V}(I)$ to be the closed sets, for ideals $I \subseteq A$.
  - The distinguished open sets $D(f)$, form a topological basis.
  - The irreducible closed sets are exactly the closures of points.
- Finally, we defined a sheaf $\mathcal{O}_X$. We set up the definition such that
  - $\mathcal{O}_X(D(f)) = A[1/f]$: at distinguished open sets $D(f)$, we get localizations too.
  - $\mathcal{O}_{X,\mathfrak{p}} = A_\mathfrak{p}$: the stalks are localizations at a prime.

  Since $D(f)$ is a basis, these two properties lets us explicitly compute $\mathcal{O}_X(U)$ for any open set $U$, so we don't have to resort to the definition using sheafification.

## §85.6 Functions are determined by germs, not values

*Prototypical example for this section: The functions* $0$ *and* $x$ *on* $\operatorname{Spec} \mathbb{C}[x]/(x^2)$.

We close the chapter with a word of warning. In any ringed space, a section is determined by its germs; so that on $\operatorname{Spec} A$ a function $f \in A$ is determined by its germ in each stalk $A_\mathfrak{p}$. However, we now will mention that an $f \in A$ is *not* determined by its value $f(\mathfrak{p}) = f \pmod{\mathfrak{p}}$ at each point.

The famous example is:

**Example 85.6.1** (On the double point, all multiples of $x$ are zero at all points)

The space $\operatorname{Spec} \mathbb{C}[x]/(x^2)$ has only one point, $(x)$. The functions $0$ and $x$ (and for that matter $2x$, $3x$, …) all vanish on it. This shows that functions are not determined uniquely by values in general.

Fortunately, we can explicitly characterize when this sort of "bad" behavior happens. Indeed, we want to see when $f(\mathfrak{p}) = g(\mathfrak{p})$ for every $\mathfrak{p}$, or equivalently, $h = f - g$ vanishes on every prime ideal $\mathfrak{p}$. This is equivalent to having

$$h \in \bigcap_{\mathfrak{p}} \mathfrak{p} = \sqrt{(0)}$$

the radical of the *zero* ideal. Thus in the prototype, the failure was caused by the fact that $x^n = 0$ for some large $n$.

**Definition 85.6.2.** For a ring $A$, the radical of the zero ideal, $\sqrt{(0)}$, is called the **nilradical** of $A$. Elements of the nilradical are called **nilpotents**. We say $A$ is **reduced** if 0 is the only nilpotent, i.e. $\sqrt{(0)} = (0)$.

> **Question 85.6.3.** Are integral domains reduced?

Then our above discussion gives:

**Theorem 85.6.4** (Nilpotents are the only issue)
Two functions $f$ and $g$ have the same value on all points of $\operatorname{Spec} A$ if and only if $f - g$ is nilpotent.

In particular, when $A$ is a reduced ring, even the values $f(\mathfrak{p})$ as $\mathfrak{p} \in \operatorname{Spec} A$ are enough to determine $f \in A$.

## §85.7 A few harder problems to think about

As Chapter 86 contains many examples of affine schemes to train your intuition; it's likely to be worth reading even before attempting these problems.

**Problem 85A**† (Spectrums are quasicompact)**.** Show that $\operatorname{Spec} A$ is quasicompact for any ring $A$.

**Problem 85B** (Punctured gyrotop, communicated by Aaron Pixton)**.** The gyrotop is the scheme $X = \operatorname{Spec} \mathbb{C}[x, y, z]/(xy, z)$. We let $U$ denote the open subset obtained by deleting the closed point $\mathfrak{m} = (x, y, z)$. Compute $\mathcal{O}_X(U)$.

**Problem 85C.** Show that a ring $R$ is a local ring if and only of the following property is true: for any $x \in R$, either $x$ or $1 - x$ is a unit.

**Problem 85D.** Let $R$ be a local ring, and $\mathfrak{m}$ be its maximal ideal. Describe $R_{\mathfrak{m}}$.

**Problem 85E.** Let $A$ be a ring, and $\mathfrak{m}$ a maximal ideal. Consider $\mathfrak{m}$ as a point of $\operatorname{Spec} A$. Show that $\kappa(\mathfrak{m}) \cong A/\mathfrak{m}$.

# 86 Interlude: eighteen examples of affine schemes

To cement in the previous two chapters, we now give an enormous list of examples. Each example gets its own section, rather than having page-long orange boxes.

One common theme you will find as you wade through the examples is that your geometric intuition may be better than your algebraic one. For example, while studying $k[x, y]/(xy)$ you will say "geometrically, I expect so-and-so to look like other thing", but when you write down the algebraic statements you find two expressions that are don't look equal to you. However, if you then do some calculation you will find that they were isomorphic after all. So in that sense, in this chapter you will learn to begin drawing pictures of algebraic statements — which is great!

As another example, all the lemmas about prime ideals from our study of localizations will begin to now take concrete forms: you will see many examples that

- $\operatorname{Spec} A/I$ looks like $\mathcal{V}(I)$ of $\operatorname{Spec} A$,
- $\operatorname{Spec} A[1/f]$ looks like $D(f)$ of $\operatorname{Spec} A$,
- $\operatorname{Spec} A_{\mathfrak{p}}$ looks like $\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}}$ of $\operatorname{Spec} A$.

In everything that follows, $k$ is any field. We will also use the following color connotations:

- The closed points of the scheme are drawn in blue.
- Non-closed points are drawn in red, with their "trails" dotted red.
- Stalks are drawn in green, when they appear.

## §86.1 Example: $\operatorname{Spec} k$, a single point

This one is easy: for any field $k$, $X = \operatorname{Spec} k$ has a single point, corresponding to the only proper ideal $(0)$. There is only way to put a topology on it.

As for the sheaf,

$$\mathcal{O}_X(X) = \mathcal{O}_{X,(0)} = k.$$

So the space is remembering what field it wants to be over. If we are complex analysts, the set of functions on a single point is $\mathbb{C}$; if we are number theorists, maybe the set of functions on a single point is $\mathbb{Q}$.

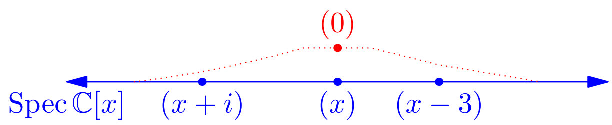
## §86.2 $\operatorname{Spec} \mathbb{C}[x]$, a one-dimensional line

The scheme $X = \operatorname{Spec} \mathbb{C}[x]$ is our beloved one-dimensional line. It consists of two types of points:

- The closed points $(x - a)$, corresponding to each complex number $a \in \mathbb{C}$, and
- The *generic* point $(0)$.

As for the Zariski topology, every open set contains $(0)$, which captures the idea it is close to everywhere: no matter where you stand, you can still hear the buzzing of the fly! True to the irreducibility of this space, the open sets are huge: the proper *closed sets* consist of finitely many closed points.

Here is a picture: for lack of better place to put it, the point $(0)$ is floating around just above the line in red.




The notion of "value at $\mathfrak{p}$" works as expected. For example, $f = x^2 + 5$ is a global section of $\mathbb{C}[x]$. If we evaluate it at $\mathfrak{p} = x - 3$, we find $f(\mathfrak{p}) = f \pmod{\mathfrak{p}} = x^2 + 5 \pmod{x - 3} = 14 \pmod{x-3}$. Indeed,

$$\kappa(\mathfrak{p}) \cong \mathbb{C}$$

meaning the stalks all have residue field $\mathbb{C}$. As

$$\mathbb{C}[x]/\mathfrak{p} \cong \mathbb{C} \qquad \text{by } x \mapsto 3$$

we see we are just plugging $x = 3$.

Of course, the stalk at $(x - 3)$ carries more information. In this case it is $\mathbb{C}[x]_{(x-3)}$. Which means that if we stand near the point $(x - 3)$, rational functions are all fine as long as no $x - 3$ appears in the denominator. So, $\frac{x^2+8}{(x-1)(x-5)}$ is a fine example of a germ near $x = 3$.

Things get more interesting if we consider the generic point $\eta = (0)$.

What is the stalk $\mathcal{O}_{X,\eta}$? Well, it should be $\mathbb{C}[x]_{(0)} = \mathbb{C}(x)$, which is the again the set of *rational* functions. And that's what you expect. For example, $\frac{x^2+8}{(x-1)(x-5)}$ certainly describes a rational function on "most" complex numbers.

What happens if we evaluate the global section $f = x^2 + 5$ at $\eta$? Well, we just get $f(\eta) = x^2 + 5$ — taking modulo 0 doesn't do much. Fitting, it means that if you want to be able to evaluate a polynomial $f$ at a general complex number, you actually just need the whole polynomial (or rational function). We can think of this in terms of the residue field being $\mathbb{C}(x)$:

$$\kappa((0)) = \operatorname{Frac}(\mathbb{C}[x]/(0)) \cong \operatorname{Frac} \mathbb{C}[x] = \mathbb{C}(x).$$

## §86.3 $\operatorname{Spec} \mathbb{R}[x]$, a one-dimensional line with complex conjugates glued (no fear nullstellensatz)

Despite appearances, this actually looks almost exactly like $\operatorname{Spec} \mathbb{C}[x]$, even more than you expect. The main thing to keep in mind is that now $(x^2+1)$ is a point, which you can loosely think of as $\pm i$. So it almost didn't matter that $\mathbb{R}$ is not algebraically closed; the $\mathbb{C}$ is showing through anyways. But this time, because we only consider real coefficient polynomials, we do not distinguish between "conjugate" $+i$ and $-i$. Put another way, we have folded $a + bi$ and $a - bi$ into a single point: $(x + i)$ and $(x - i)$ merge to form $x^2 + 1$.

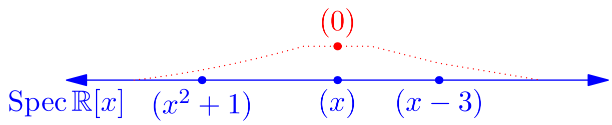
To be explicit, there are three types of points:

- $(x - a)$ for each real number $a$
- $(x^2 - ax + b)$ if $a^2 < 4b$, and

- the generic point $(0)$, again.

The ideals $(x-a)$ and $(x^2-ax+b)$ are each closed points: the quotients with $\mathbb{R}[x]$ are both fields ($\mathbb{R}$ and $\mathbb{C}$, respectively).

We have been drawing $\operatorname{Spec}\mathbb{C}[x]$ as a one-dimensional line, so $\operatorname{Spec}\mathbb{R}[x]$ will be drawn the same way.




One nice thing about this is that the nullstellensatz is less scary than it was with classical varieties. The short version is that the function $x^2+1$ vanishes at a point of $\operatorname{Spec}\mathbb{R}[x]$, namely $(x^2+1)$ itself! (So in some ways we're sort of automatically working with the algebraic closure.)

You might remember a long time ago we made a big fuss about the weak nullstellensatz, for example in Problem 77C: if $I$ was a proper ideal in $\mathbb{C}[x_1,\dots,x_n]$ there was *some* point $(a_1,\dots,a_n)\in\mathbb{C}^n$ such that $f(a_1,\dots,a_n)=0$ for all $f\in I$. With schemes, it doesn't matter anymore: if $I$ is a proper ideal of a ring $A$, then some maximal ideal contains it, and so $\mathcal{V}(I)$ is nonempty in $\operatorname{Spec} A$.

We better mention that the stalks this time look different than expected. Here are some examples:

$$\begin{aligned}\kappa\left((x^2+1)\right)&\cong\mathbb{R}[x]/(x^2+1)\cong\mathbb{C}\\ \kappa\left((x-3)\right)&\cong\mathbb{R}[x]/(x-3)\cong\mathbb{R}\\ \kappa\left((0)\right)&\cong\operatorname{Frac}(\mathbb{R}[x]/(0))\cong\mathbb{R}(x)\end{aligned}$$

Notice the residue fields above the "complex" points are bigger: functions on them take values in $\mathbb{C}$.

## §86.4 $\operatorname{Spec} k[x]$, over any ground field

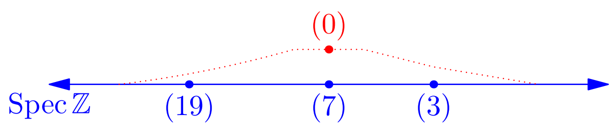
In general, if $\overline{k}$ is the algebraic closure of $k$, then $\operatorname{Spec} k[x]$ looks like $\operatorname{Spec}\overline{k}[x]$ with all the Galois conjugates glued together. So we will almost never need "algebraically closed" hypotheses anymore: we're working with polynomial ideals, so all the elements are implicitly there, anyways.

## §86.5 $\operatorname{Spec}\mathbb{Z}$, a one-dimensional scheme

The great thing about $\operatorname{Spec}\mathbb{Z}$ is that it basically looks like $\operatorname{Spec} k[x]$, too, being a one-dimensional scheme. It has two types of prime ideals:

- $(p)$, for every rational prime $p$,
- and the generic point $(0)$.

So the picture almost does not change.



This time $\eta = (0)$ has stalk $\mathbb{Z}_{(0)} = \mathbb{Q}$, so a "rational function" is literally a rational number! Thus, $\frac{20}{19}$ is a function with a double root at $(2)$, a root at $(5)$, and a simple pole at $(19)$. If we evaluate it at $\mathfrak{p} = (7)$, we get $3 \pmod 7$. In general, the residue fields are what you'd guess:

$$\kappa((p)) = \mathbb{Z}/(p) \cong \mathbb{F}_p$$

for each prime $p$, and $\kappa((0)) \cong \mathbb{Q}$.

The stalk is bigger than the residue field at the closed points: for example

$$\mathcal{O}_{\operatorname{Spec}\mathbb{Z},(3)} \cong \left\{ \frac{m}{n} \mid 3 \nmid n \right\}$$

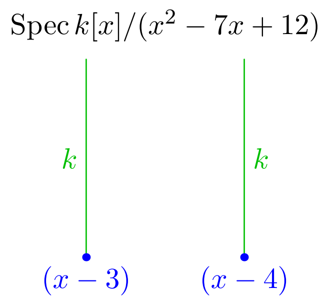
consists of rational numbers with no pole at 3. The stalk at the generic point is $\mathbb{Z}_{(0)} \cong \operatorname{Frac}\mathbb{Z} = \mathbb{Q}$.

## §86.6 $\operatorname{Spec} k[x]/(x^2 - 7x + 12)$, two points

If we were working with affine varieties, you would already know what the answer is: $x^2 - 7x + 12 = 0$ has solutions $x = 3$ and $x = 4$, so this should be a scheme with two points.

To see this come true, we use Proposition 83.6.1: the points of $\operatorname{Spec} k[x]/(x^2 - 7x + 12)$ should correspond to prime ideals of $k[x]$ containing $(x^2 - 7x + 12)$. As $k[x]$ is a PID, there are only two, $(x - 3)$ and $(x - 4)$. They are each maximal, since their quotient with $k[x]$ is a field (namely $k$), so as promised $\operatorname{Spec} k[x]/(x^2 - 7x + 12)$ has just two closed points.

Each point has a stalk above it isomorphic to $k$. A section on the whole space $X$ is just a choice of two values, one at $(x - 3)$ and one at $(x - 4)$.




So actually, this is a geometric way of thinking about the ring-theoretic fact that

$$k[x]/\left(x^2 - 7x + 12\right) \cong k \times k \quad \text{by } f \mapsto (f(3), f(4)).$$

Also, this is the first example of a reducible space in this chapter: in fact $X$ is even disconnected. Accordingly there is no generic point floating around: as the space is discrete, all points are closed.

## §86.7 $\operatorname{Spec} k[x]/(x^2)$, the double point

We can now elaborate on the "double point" scheme

$$X_2 = \operatorname{Spec} k[x]/(x^2)$$

since it is such an important motivating example. How it does differ from the "one-point" scheme $X_1 = \operatorname{Spec} k[x]/(x) = \operatorname{Spec} k$? Both $X_2$ and $X_1$ have exactly one point, and so obviously the topologies are the same too.

The difference is that the stalk (equivalently, the section, since we have only one point) is larger:

$$\mathcal{O}_{X_2,(x)} = \mathcal{O}_{X_2}(X_2) = k[x]/(x^2).$$

So to specify a function on a double point, you need to specify two parameters, not just one: if we take a polynomial

$$f = a_0 + a_1 x + \cdots \in k[x]$$

then evaluating it at the double point will remember both $a_0$ and the "first derivative" say.

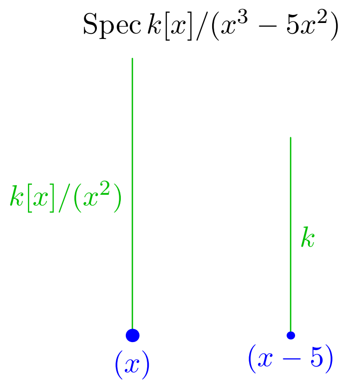
I should mention that if you drop all the way to the residue fields, you can't tell the difference between the double point and the single point anymore. For the residue field of $\operatorname{Spec} k[x]/(x^2)$ at $(x)$ is

$$\operatorname{Frac}(A/(x)) = \operatorname{Frac} k = k.$$

Thus the set of *values* is still just $k$ (leading to the "nilpotent" discussion at the end of last chapter); but the stalk, having "enriched" values, can tell the difference.

## §86.8 $\operatorname{Spec} k[x]/(x^3 - 5x^2)$, a double point and a single point

There is no problem putting the previous two examples side by side: the scheme $X = \operatorname{Spec} k[x]/(x^3 - 5x^2)$ consists of a double point next to a single point. Note that the stalks are different: the one above the double point is larger.




This time, we implicitly have the ring isomorphism

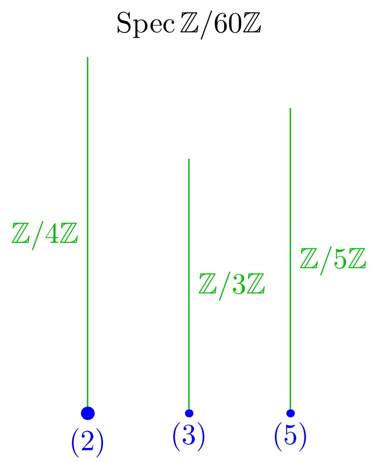
$$k[x]/(x^3 - 5x^2) \cong k[x]/(x^2) \times k$$

by $f \mapsto (f(0) + f'(0)x, f(5))$. The derivative is meant formally here!

## §86.9 $\operatorname{Spec} \mathbb{Z}/60\mathbb{Z}$, a scheme with three points

We've being seeing geometric examples of ring products coming up, but actually the Chinese remainder theorem you are used to with integers is no different. (This example $X = \operatorname{Spec} \mathbb{Z}/60\mathbb{Z}$ is taken from [**Va17**, §4.4.11].)

By Proposition 83.6.1, the prime ideals of $\mathbb{Z}/60\mathbb{Z}$ are $(2)$, $(3)$, $(5)$. But you can think of this also as coming out of $\operatorname{Spec} \mathbb{Z}$: as 60 was a function with a double root at $(2)$, and single roots at $(3)$ and $(5)$.




Actually, although I have been claiming the ring isomorphisms, the sheaves really actually give us a full proof. Let me phrase it in terms of global sections:

$$\begin{aligned}\mathbb{Z}/60\mathbb{Z} &= \mathcal{O}_X(X) \\ &= \mathcal{O}_X(\{(2)\}) \times \mathcal{O}_X(\{(3)\}) \times \mathcal{O}_X(\{(5)\}) \\ &= \mathcal{O}_{X,(2)} \times \mathcal{O}_{X,(3)} \times \mathcal{O}_{X,(5)} \\ &= \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}.\end{aligned}$$

So the theorem that $\mathcal{O}_X(X) = A$ for $X = \operatorname{Spec} A$ is doing the "work" here; the sheaf axioms then give us the Chinese remainder theorem from here.

On that note, this gives us a way of thinking about the earlier example that

$$(\mathbb{Z}/60\mathbb{Z})[1/5] \cong \mathbb{Z}/12\mathbb{Z}.$$

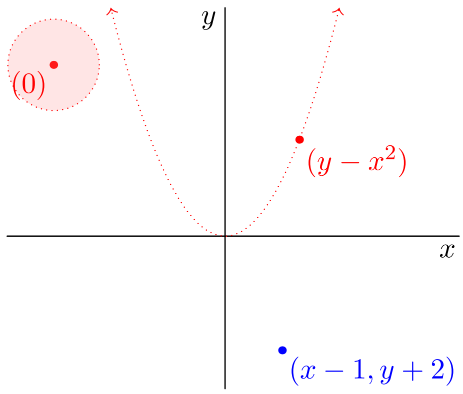
Indeed, $\operatorname{Spec} \mathbb{Z}/60\mathbb{Z}[1/5]$ is supposed to look like the distinguished open set $D(5)$: which means we delete the point $(5)$ from the picture above. That leaves us with $\mathbb{Z}/12\mathbb{Z}$.

## §86.10 $\operatorname{Spec} k[x,y]$, the two-dimensional plane

We have seen this scheme already: it is visualized as a plane. There are three types of points:

- The closed points $(x-a, y-b)$, which consists of single points of the plane.
- A non-closed point $(f(x,y))$ for any irreducible polynomial $f$, which floats along some irreducible curve. We illustrate this by drawing the dotted curve along which the point is floating.
- The generic point $(0)$, floating along the entire plane. I don't know a good place to put it in the picture, so I'll just put it somewhere and draw a dotted circle around it.

Here is an illustration of all three types of points.




We also go ahead and compute the stalks above each point.

- The stalk above $(x - 1, y + 2)$ is the set of rational functions $\frac{f(x,y)}{g(x,y)}$ such that $g(1, -2) \neq 0$.

- The stalk above the non-closed point $(y - x^2)$ is the set of rational functions $\frac{f(x,y)}{g(x,y)}$ such that $g(t, t^2) \neq 0$. For example the function $\frac{xy}{x+y-2}$ is still fine; despite the fact that the denominator vanishes at the point $(1, 1)$ and $(-2, 4)$ on the parabola, it is a function on a "generic point" (crudely, "most points") of the parabola.

- The stalk above $(0)$ is the entire fraction field $k(x, y)$ of rational functions.

Let's consider the global section $f = x^2 + y^2$ and also take the value at each of the above points.

- $f \pmod{x - 1, y - 2} = 5$, so $f$ has value 5 at $(x - 1, y + 2)$.

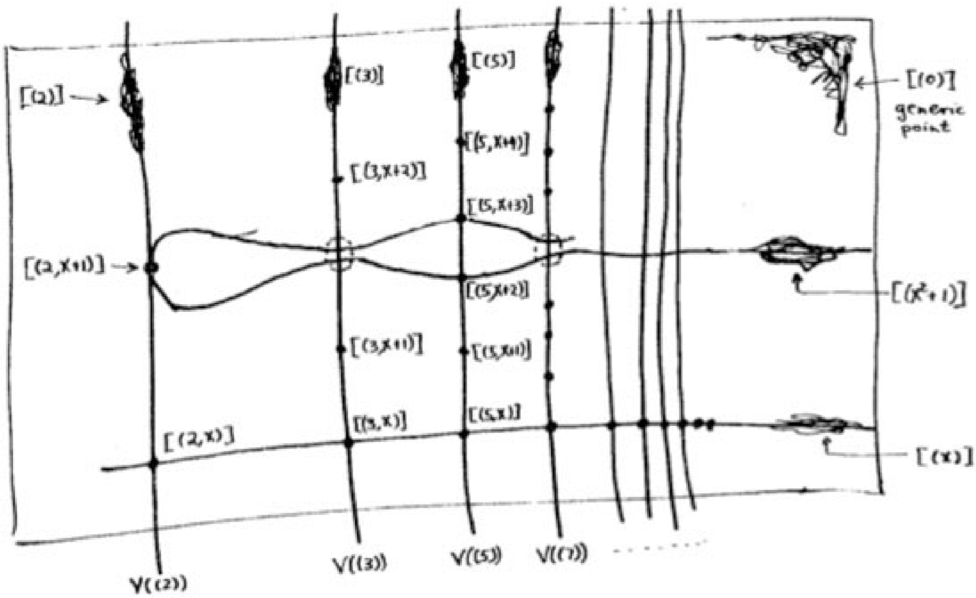
- The new bit is that we can think of evaluating $f$ along the parabola too — it is given a particular value in the quotient $k[x, y]/(y - x^2)$. We can think of it as $f = x^2 + y^2 \equiv x^2 + x^4 \pmod{y - x^2}$ for example. Note that if we know the value of $f$ at the generic point of the parabola, we can therefore also evaluate it at any closed point on the parabola.

- At the generic point $(0)$, $f \pmod 0 = f$. So "evaluating at the generic point" does nothing, as in any other scheme.

## §86.11 $\operatorname{Spec} \mathbb{Z}[x]$, a two-dimensional scheme, and Mumford's picture

We saw $\operatorname{Spec} \mathbb{Z}$ looked a lot like $\operatorname{Spec} k[x]$, and we will now see that $\operatorname{Spec} \mathbb{Z}[x]$ looks a lot like $\operatorname{Spec} k[x, y]$.

There is a famous picture of this scheme in Mumford's "red book", which I will produce here for culture-preservation reasons, even though there are some discrepancies between the pictures that we previously drew.




Mumford uses $[\mathfrak{p}]$ to denote the point $\mathfrak{p}$, which we don't, so you can ignore the square brackets that appear everywhere. The non-closed points are illustrated as balls of fuzz.

As before, there are three types of prime ideals, but they will look somewhat more different:

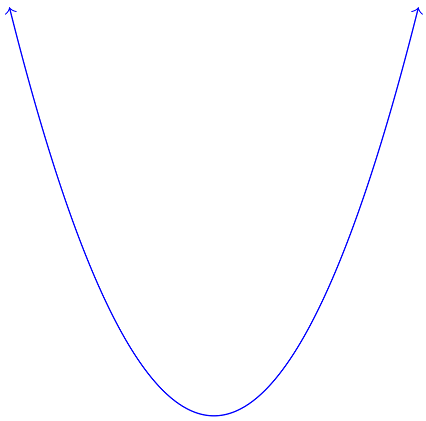
- The closed points are now pairs $(p, f(x))$ where $p$ is a prime and $f$ is an irreducible polynomial modulo $p$. Indeed, these are the maximal ideals: the quotient $\mathbb{Z}[x]/(p, f)$ becomes some finite extension of $\mathbb{F}_p$.
- There are now two different "one-dimensional" non-closed points:
  - Each rational prime gives a point $(p)$ and
  - Each irreducible polynomial $f$ gives a point $(f)$.

  Indeed, note that the quotients of $\mathbb{Z}[x]$ by each are integral domains.
- $\mathbb{Z}[x]$ is an integral domain, so as always $(0)$ is our generic point for the entire space.

There is one bit that I would do differently, in $\mathcal{V}(3)$ and $\mathcal{V}(7)$, there ought to be a point $(3, x^2+1)$, which is not drawn as a closed point in the picture, but rather as dashed oval. This is not right in the topological sense: as $\mathfrak{m} = (3, x^2+1)$ is a maximal ideal, so it really is one closed point in the scheme. But the reason it might be thought of as "doubled", is that $\mathbb{Z}[x]/(3, x^2+1)$, the residue field at $\mathfrak{m}$, is a two-dimensional $\mathbb{F}_3$ vector space.

## §86.12 Spec $k[x,y]/(y-x^2)$, the parabola

By Proposition 83.6.1, the prime ideals of $k[x,y]/(y-x^2)$ correspond to the prime ideals of $k[x,y]$ which are supersets of $(y-x^2)$, or equivalently the points of $\operatorname{Spec} k[x,y]$ contained inside the closed set $\mathcal{V}(y-x^2)$. Moreover, the subspace topology on $\mathcal{V}(y-x^2)$ coincides with the topology on $\operatorname{Spec} k[x,y]/(y-x^2)$.



Spec $k[x,y]/(y-x^2)$

This holds much more generally:

**Exercise 86.12.1** (Boring check)**.** Show that if $I$ is an ideal of a ring $A$, then Spec $A/I$ is homeomorphic as a topological space to the closed subset $\mathcal{V}(I)$ of Spec $A$.

So this is the notion of "closed embedding": the parabola, which was a closed subset of Spec $k[x,y]$, is itself a scheme. It will be possible to say more about this, once we actually define the notion of a morphism.

The sheaf on this scheme only remembers the functions on the parabola, though: the stalks are not "inherited", so to speak. To see this, let's compute the stalk at the origin: Theorem 83.7.1 tells us it is

$$k[x,y]_{(x,y)}/(y-x^2) \cong k[x,x^2]_{(x,x^2)} \cong k[x]_{(x)}$$

which is the same as the stalk of the affine line Spec $k[x]$ at the origin. Intuitively, not surprising; if one looks at any point of the parabola near the origin, it looks essentially like a line, as do the functions on it.

The stalk above the generic point is $\operatorname{Frac}(k[x,y]/(y-x^2))$: so rational functions, with the identification that $y = x^2$. Also unsurprising.

Finally, we expect the parabola is actually isomorphic to Spec $k[x]$, since there is an isomorphism $k[x,y]/(y-x^2) \cong k[x]$ by sending $y \mapsto x^2$. Pictorially, this looks like "un-bending" the hyperbola. In general, we would hope that when two rings $A$ and $B$ are isomorphic, then Spec $A$ and Spec $B$ should be "the same" (otherwise we would be horrified), and we'll see later this is indeed the case.

## §86.13 Spec $\mathbb{Z}[i]$, the Gaussian integers (one-dimensional)

You can play on this idea some more in the integer case. Note that

$$\mathbb{Z}[i] \cong \mathbb{Z}[x]/(x^2+1)$$

which means this is a "dimension-one" closed set within Spec $\mathbb{Z}[x]$. In this way, we get a scheme whose elements are *Gaussian primes.*

You can tell which closed points are "bigger" than others by looking at the residue fields. For example the residue field of the point $(2+i)$ is

$$\kappa\left((2+i)\right) = \mathbb{Z}[i]/(2+i) \cong \mathbb{F}_5$$

but the residue field of the point (3) is

$$\kappa\left((3)\right) \cong \mathbb{Z}[i]/(3) \cong \mathbb{F}_9$$

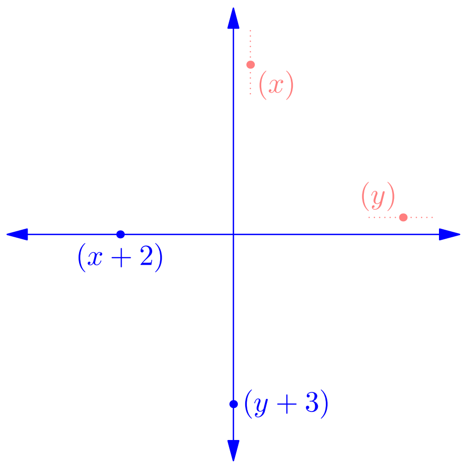
which is a degree two $\mathbb{F}_3$-extension.

## §86.14 Long example: $\operatorname{Spec} k[x,y]/(xy)$, two axes

This is going to be our first example of a non-irreducible scheme.

### §86.14.i Picture

Like before, topologically it looks like the closed set $\mathcal{V}(xy)$ of $\operatorname{Spec} k[x,y]$. Here is a picture:




To make sure things are making sense:

> **Question 86.14.1** (Sanity check)**.** Verify that $(y+3)$ is really a maximal ideal of $\operatorname{Spec} k[x,y]/(xy)$ lying in $\mathcal{V}(x)$.

The ideal (0) is longer prime, so it is not a point of this space. Rather, there are two non-closed points this time: the ideals $(x)$ and $(y)$, which can be visualized as floating around each of the two axes. This space is reducible, since it can be written as the union of two proper closed sets, $\mathcal{V}(x) \cup \mathcal{V}(y)$. (It is still *connected*, as a topological space.)

### §86.14.ii Throwing out the $y$-axis

Consider the distinguished open set $U = D(x)$. This corresponds to deleting $\mathcal{V}(x)$, the $y$-axis. Therefore we expect that $D(x)$ "is" just $\operatorname{Spec} k[x]$ with the origin deleted, and in particular that we should get $k[x, x^{-1}]$ for the sections. Indeed,

$$\begin{aligned}\mathcal{O}_{\operatorname{Spec} k[x,y]/(xy)}(D(x)) &\cong (k[x,y]/(xy))[1/x] \\ &\cong k[x,x^{-1},y]/(xy) \cong k[x,x^{-1},y]/(y) \cong k[x,x^{-1}].\end{aligned}$$

where $(xy) = (y)$ follows from $x$ being a unit. Everything as planned.

### §86.14.iii Stalks above some points

Let's compute the stalk above the point $\mathfrak{m} = (x+2)$, which we think of as the point $(-2,0)$ on the $x$-axis. (If it makes you more comfortable, note that $\mathfrak{m} \ni y(x+2) = 2y$ and hence $y \in \mathfrak{m}$, so we could also write $\mathfrak{m} = (x+2, y)$.) The stalk is

$$\mathcal{O}_{\operatorname{Spec} k[x,y]/(xy), \mathfrak{m}} = (k[x,y]/(xy))_{(x+2)}.$$

But I claim that $y$ becomes the zero element with this localization. Indeed, we have $\frac{y}{1} = \frac{0}{x} = 0$. Hence the entire thing collapses to just

$$\mathcal{O}_{\operatorname{Spec} k[x,y]/(xy), \mathfrak{m}} = k[x]_{(x+2)}$$

which anyways is the stalk of $(x+2)$ in $\operatorname{Spec} k[x]$. That's expected. If we have a space with two lines but we're standing away from the origin, then the stalk is not going to pick up the weird behavior at that far-away point; it only cares about what happens near $\mathfrak{m}$, and so it looks just like an affine line there.

> **Remark 86.14.2** — Note that $(k[x,y]/(xy))_{(x+2)}$ is *not* the same as $k[x,y]_{(x+2)}/(xy)$; the order matters here. In fact, the latter is the zero ring, since both $x$ and $y$, and hence $xy$, are units.

The generic point $(y)$ (which floats around the $x$-axis) will tell a similar tale: if we look at the stalk above it, we ought to find that it doesn't recognize the presence of the $y$-axis, because "nearly all" points don't recognize it either. To actually compute the stalk:

$$\mathcal{O}_{\operatorname{Spec} k[x,y]/(xy), (y)} = (k[x,y]/(xy))_{(y)}.$$

Again $\frac{y}{1} = \frac{0}{x} = 0$, so this is just

$$\mathcal{O}_{\operatorname{Spec} k[x,y]/(xy), (y)} \cong k[x]_{(0)} \cong k(x).$$

which is what we expected (it is the same as the stalk above $(0)$ in $\operatorname{Spec} k[x]$).

### §86.14.iv Stalk above the origin (tricky)

The stalk above the origin $(x,y)$ is interesting, and has some ideas in it we won't be able to explore fully without talking about localizations of modules. The localization is given by

$$(k[x,y]/(xy))_{(x,y)}$$

and hence the elements should be

$$\frac{c + (a_1 x + a_2 x^2 + \dots) + (b_1 y + b_2 y^2 + \dots)}{c' + (a_1' x + a_2' x^2 + \dots) + (b_1' y + b_2' y^2 + \dots)}$$

where $c' \neq 0$.

You might feel unsatisfied with this characterization. Here is some geometric intuition. You can write the global section ring as

$$k[x,y]/(xy) = c + (a_1 x + a_2 x^2 + \dots) + (b_1 y + b_2 y^2 + \dots)$$

meaning any global section is the sum of an $x$-polynomial and a $y$-polynomial. This is *not* just the ring product $k[x] \times k[y]$, though; the constant term is shared. So it's better thought of as pairs of polynomials in $k[x]$ and $k[y]$ which agree on the constant term.

If you like category theory, it is thus a fibered product

$$k[x,y]/(xy) \cong k[x] \times_k k[y]$$

with morphism $k[x] \to k$ and $k[y] \to k$ by sending $x$ and $y$ to zero. In that way, we can mostly decompose $k[x,y]/(xy)$ into its two components.

We really ought to be able to do the same as the stalk: we wish to say that

$$\mathcal{O}_{\operatorname{Spec} k[x,y]/(xy),(x,y)} \cong k[x]_{(x)} \times_k k[y]_{(y)}.$$

English translation: a "typical" germ ought to look like $\frac{3+x}{x^2+7} + \frac{4+y^3}{y^2+y+7}$, with the $x$ and $y$ parts decoupled. Equivalently, the stalk should consist of pairs of $x$-germs and $y$-germs that agree at the origin.

In fact, this is true! This might come as a surprise, but let's see why we expect this. Suppose we take the germ

$$\frac{1}{1-(x+y)}.$$

If we hold our breath, we could imagine expanding it as a geometric series: $1 + (x+y) + (x+y)^2 + \dots$. As $xy = 0$, this just becomes $1 + x + x^2 + x^3 + \dots + y + y^2 + y^3 + \dots$. This is nonsense (as written), but nonetheless it suggests the conjecture

$$\frac{1}{1-(x+y)} = \frac{1}{1-x} + \frac{1}{1-y} - 1$$

which you can actually verify is true.

> **Question 86.14.3.** Check this identity holds.

Of course, this is a lot of computation just for one simple example. Is there a way to make it general? Yes: the key claim is that "localization commutes with *limits*". You can try to work out the statement now if you want, but we won't do so.

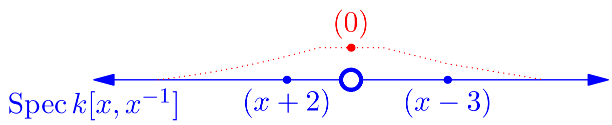
## §86.15 $\operatorname{Spec} k[x, x^{-1}]$, the punctured line (or hyperbola)

This is supposed to look like $D(x)$ of $\operatorname{Spec} k[x]$, or the line with the origin deleted it. Alternatively, we could also write

$$k[x, x^{-1}] \cong k[x,y]/(xy-1)$$

so that the scheme could also be drawn as a hyperbola.

First, here's the 1D illustration.

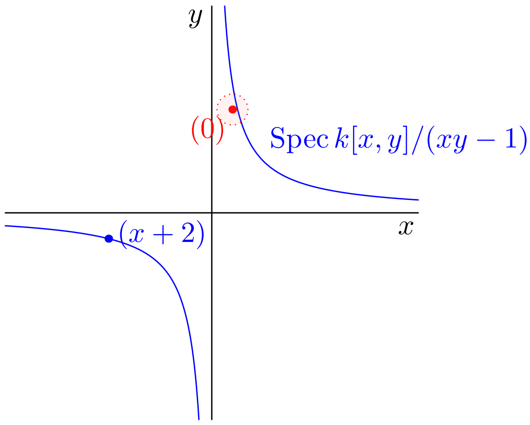



We actually saw this scheme already when we took $\operatorname{Spec} k[x,y]/(xy)$ and looked at $D(y)$, too. Anyways, let us compute the stalk at $(x-3)$ now; it is

$$\mathcal{O}_{\operatorname{Spec} k[x,x^{-1}],(x-3)} \cong k[x,x^{-1}]_{(x-3)} \cong k[x]_{(x-3)}$$

since $x^{-1}$ is in $k[x]_{(x-3)}$ anyways. So again, we see that the deletion of the origin doesn't affect the stalk at the farther away point $(x-3)$.

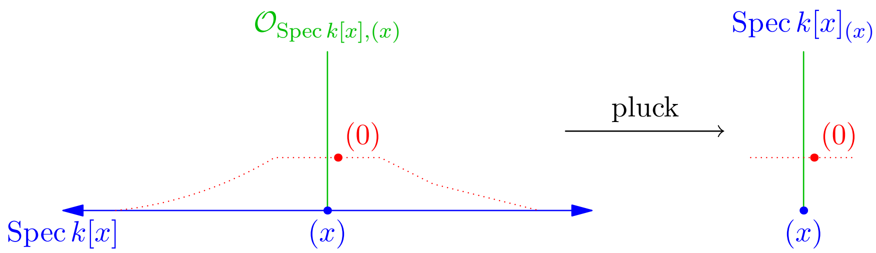
As mentioned, since $k[x, x^{-1}]$ is isomorphic to $k[x, y]/(xy - 1)$, another way to draw the visualize the same curve would be to draw the hyperbola (which you can imagine as flattening to give the punctured line.) There is one generic point $(0)$ since $k[x, y]/(xy - 1)$ really is an integral domain, as well as points like $(x + 2, y + 1/2) = (x + 2) = (y + 1/2)$.




## §86.16 $\operatorname{Spec} k[x]_{(x)}$, zooming in to the origin of the line

We know already that $\mathcal{O}_{\operatorname{Spec} A, \mathfrak{p}} \cong A_\mathfrak{p}$: so $A_\mathfrak{p}$ should be the stalk at $\mathfrak{p}$. In this example we will see that $\operatorname{Spec} A_\mathfrak{p}$ should be drawn sort of as this stalk, too.

We saw earlier how to draw a picture of $\operatorname{Spec} k[x]$. You can also draw a picture of the stalk above the origin $(x)$, which you might visualize as a grass or other plant growing above $(x)$ if you like agriculture. In that case, $\operatorname{Spec} k[x]_{(x)}$ might look like what happens if you pluck out that stalk from the affine line.




Since $k[x]_{(x)}$ is a local ring (it is the localization of a prime ideal), this point has only one closed point: the maximal ideal $(x)$. However, surprisingly, it has one more point: a "generic" point $(0)$. So $\operatorname{Spec} k[x]_{(x)}$ is a *two-point space*, but it does not have the discrete topology: $(x)$ is a closed point, but $(0)$ is not. (This makes it a nice counter-example for exercises of various sorts.)

So, topologically what's happening is that when we zoom in to $(x)$, the generic point $(0)$ (which was "close to every point") remains, floating above the point $(x)$.

Note that the stalk above our single closed point $(x)$ is the same as it was before:

$$\left(k[x]_{(x)}\right)_{(x)} \cong k[x]_{(x)}.$$

Indeed, in general if $R$ is a local ring with maximal ideal $\mathfrak{m}$, then $R_\mathfrak{m} \cong R$: since every element $x \notin \mathfrak{m}$ was invertible anyways. Thus in the picture, the stalk is drawn the same.

Similarly, the stalk above $(0)$ is the same as it was before we plucked it out:

$$\left(k[x]_{(x)}\right)_{(0)} = \operatorname{Frac} k[x]_{(x)} = k(x).$$

More generally:

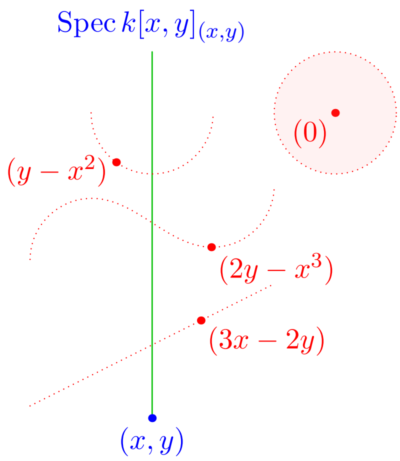
**Exercise 86.16.1.** Let $A$ be a ring, and $\mathfrak{q} \subseteq \mathfrak{p}$ prime ideals. Check that $A_\mathfrak{q} \cong (A_\mathfrak{p})_\mathfrak{q}$, where we view $\mathfrak{q}$ as a prime ideal of $A_\mathfrak{p}$.

So when we zoom in like this, all the stalks stay the same, even above the non-closed points.

## §86.17 $\operatorname{Spec} k[x,y]_{(x,y)}$, zooming in to the origin of the plane

The situation is more surprising if we pluck the stalk above the origin of $\operatorname{Spec} k[x,y]$, the two-dimensional plane. The points of $\operatorname{Spec} k[x,y]_{(x,y)}$ are supposed to be the prime ideals of $k[x,y]$ which are contained in $(x,y)$; geometrically these are $(x,y)$ and the generic points passing through the origin. For example, there will be a generic point for the parabola $(y-x^2)$ contained in $k[x,y]_{(x,y)}$, and another one $(y-x)$ corresponding to a straight line, etc.

So we have the single closed point $(x,y)$ sitting at the bottom, and all sorts of "one-dimensional" generic points floating above it: lines, parabolas, you name it. Finally, we have $(0)$, a generic point floating in two dimensions, whose closure equals the entire space.




## §86.18 $\operatorname{Spec} k[x,y]_{(0)} = \operatorname{Spec} k(x,y)$, the stalk above the generic point

The generic point of the plane just has stalk $\operatorname{Spec} k(x,y)$: which is the spectrum of a field, hence a single point. The stalk remains intact as compared to when planted in $\operatorname{Spec} k[x,y]$; the functions are exactly rational functions in $x$ and $y$.

## §86.19 A few harder problems to think about

**Problem 86A.** Draw a picture of $\operatorname{Spec} \mathbb{Z}[1/55]$, describe the topology, and compute the stalk at each point.

**Problem 86B.** Draw a picture of $\operatorname{Spec} \mathbb{Z}_{(5)}$, describe the topology, and compute the stalk at each point.

**Problem 86C.** Let $A = (k[x,y]/(xy))[(x+y)^{-1}]$. Draw a picture of $\operatorname{Spec} A$. Show that it is not connected as a topological space.

**Problem 86D.** Let $A = k[x,y]_{(y-x^2)}$. Draw a picture of $\operatorname{Spec} A$.

# 87 Morphisms of locally ringed spaces

Having set up the definition of a locally ringed space, we are almost ready to define morphisms between them. Throughout this chapter, you should imagine your ringed spaces are the affine schemes we have so painstakingly defined; but it will not change anything to work in the generality of arbitrary locally ringed spaces.

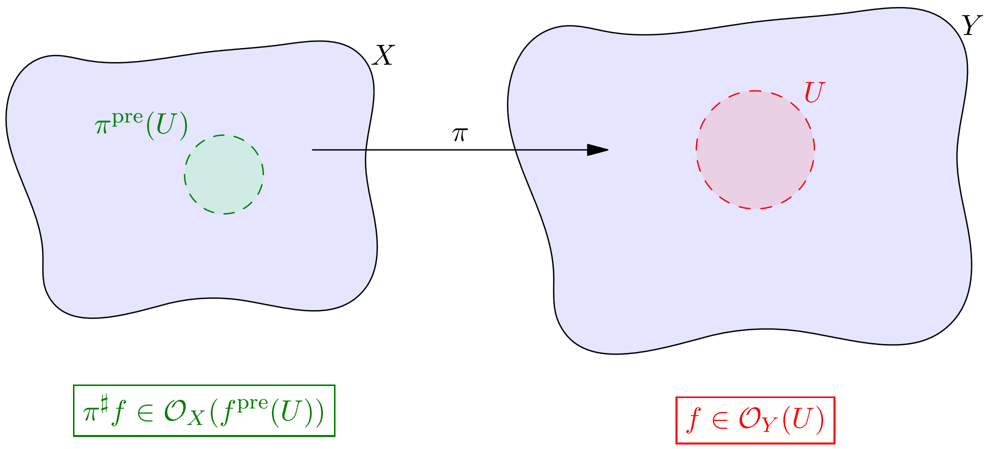
## §87.1 Morphisms of ringed spaces via sections

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces. We want to give a define what it means to have a function $\pi\colon X \to Y$ between them.[1] We start by requiring the map to be continuous, but this is not enough: there is a sheaf on it!

Well, you might remember what we did for baby ringed spaces: any time we had a function on an open set of $U \subseteq Y$, we wanted there to be an analogous function on $\pi^{\text{pre}}(U) \subseteq X$. For baby ringed spaces, this was done by composition, since the elements of the sheaf *were* really complex valued functions:

$$\pi^\sharp \phi \text{ was defined as } \phi \circ \pi.$$

The upshot was that we got a map $\mathcal{O}_Y(U) \to \mathcal{O}_X(\pi^{\text{pre}}(U))$ for every open set $U$.




Now, for general locally ringed spaces, the sections are just random rings, which may not be so well-behaved. So the solution is that we *include* the data of $f^\sharp$ as part of the definition of a morphism.

**Remark 87.1.1** — As we will see in Example 87.4.2, unlike the situation in algebraic varieties where the morphism is uniquely determined by the map of topological space, here $\pi^\sharp$ is not necessarily uniquely determined by the map $\pi$. Thus, including the $\pi^\sharp$ is necessary.

**Definition 87.1.2.** A **morphism of ringed spaces** $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a pair $(\pi, \pi^\sharp)$ where $\pi\colon X \to Y$ is a continuous map (of topological spaces), and $\pi^\sharp$ consists of a choice of ring homomorphism

$$\pi^\sharp_U\colon \mathcal{O}_Y(U) \to \mathcal{O}_X(\pi^{\text{pre}}(U))$$

[1] Notational connotations: for ringed spaces, $\pi$ will be used for maps, since $f$ is often used for sections.

for every open set $U \subseteq Y$, such that the restriction diagram

$$\begin{tikzcd}
\mathcal{O}_Y(U) \arrow[r] \arrow[d] & \mathcal{O}_X(\pi^{\text{pre}}(U)) \arrow[d] \\
\mathcal{O}_Y(V) \arrow[r] & \mathcal{O}_X(\pi^{\text{pre}}(V))
\end{tikzcd}$$

commutes for $V \subseteq U$.

**Abuse of Notation 87.1.3.** We will abbreviate $(\pi, \pi^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ to just $\pi\colon X \to Y$, despite the fact this notation is exactly the same as that for topological spaces.

There is an obvious identity map, and so we can also define isomorphism etc. in the categorical way.

## §87.2 Morphisms of ringed spaces via stalks

Unsurprisingly, the sections are clumsier to work with than the stalks, now that we have grown to love localization. So rather than specifying $\pi^\sharp_U$ on every open set $U$, it seems better if we could do it by stalks (there are fewer stalks than open sets, so this saves us a lot of work!).

We start out by observing that we *do* get a morphism of stalks.

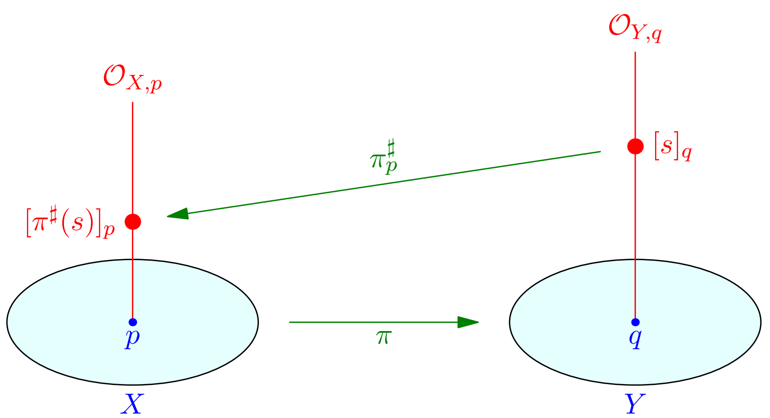
**Proposition 87.2.1** (Induced stalk morphisms)

If $\pi\colon X \to Y$ is a map of ringed spaces sending $\pi(p) = q$, then we get a map

$$\pi^\sharp_p\colon \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$$

whenever $\pi(p) = q$.

This means you can draw a morphism of locally ringed spaces as a continuous map on the topological space, plus for each $\pi(p) = q$, an assignment of each germ at $q$ to a germ at $p$.




Again, compare this to the pullback picture: this is roughly saying that if a function $f$ has some enriched value at $q$, then $\pi^\sharp(f)$ should be assigned a corresponding enriched value at $p$. The analogy is not perfect since the stalks at $q$ and $p$ may be different rings in general, but there should at least be a ring homomorphism (the assignment).

*Proof.* If $(s, U)$ is a germ at $q$, then $(\pi^\sharp(s), \pi^{\text{pre}}(U))$ is a germ at $p$, and this is a well-defined morphism because of compatibility with restrictions. □

We already obviously have uniqueness in the following senes.

**Proposition 87.2.2** (Uniqueness of morphisms via stalks)
Consider a map of ringed spaces $(\pi, \pi^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and the corresponding map $\pi^\sharp_p$ of stalks. Then $\pi^\sharp$ is uniquely determined by $\pi^\sharp_p$.

*Proof.* Given a section $s \in \mathcal{O}_Y(U)$, let

$$t = \pi^\sharp_U(s) \in \mathcal{O}_X(\pi^{\text{pre}}(U))$$

denote the image under $\pi^\sharp$.

We know $t_p$ for each $p \in \pi^{\text{pre}}(U)$, since it equals $\pi^\sharp_p(t)$ by definition. That is, we know all the germs of $t$. So we know $t$. □

However, it seems clear that not every choice of stalk morphisms will lead to $\pi^\sharp_U$: some sort of "continuity" or "compatibility" is needed. You can actually write down the explicit statement: each sequence of compatible germs over $U$ should get mapped to a sequence of compatible germs over $\pi^{\text{pre}}(U)$. We avoid putting up a formal statement of this for now, because the statement is clumsy, and you're about to see it in practice (where it will make more sense).

**Remark 87.2.3** (Isomorphisms are determined by stalks) — One fact worth mentioning, that we won't prove, but good to know: a map of ringed spaces $(\pi, \pi^\sharp)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is an isomorphism if and only if $\pi$ is a homeomorphism, and moreover $\pi^\sharp_p$ is an isomorphism for each $p \in X$.

## §87.3 Morphisms of locally ringed spaces

On the other hand, we've seen that our stalks are local rings, which enable us to actually talk about *values*. And so we want to add one more compatibility condition to ensure that our notion of value is preserved. Now the stalks at $p$ and $q$ in the previous picture might be different, so $\kappa(p)$ and $\kappa(q)$ might even be different fields.

**Definition 87.3.1.** A **morphism of locally ringed spaces** is a morphism of ringed spaces $\pi\colon X \to Y$ with the following additional property: whenever $\pi(p) = q$, the map at the stalks also induces a well-defined ring homomorphism

$$\pi^\sharp_p\colon \kappa(q) \to \kappa(p).$$

So we require $\pi^\sharp_p$ induces a field homomorphism[2] on the *residue fields*. In particular, since $\pi^\sharp(0) = 0$, this means something very important:

**In a morphism of locally ringed spaces, a germ vanishes at $q$ if and only if the corresponding germ vanishes at $p$.**

[2]Which means it is automatically injective, by Problem 5B.

**Exercise 87.3.2** (So-called "local ring homomorphism"). Show that this is equivalent to requiring
$$(\pi_p^\sharp)^{\text{img}}(\mathfrak{m}_{Y,q}) \subseteq \mathfrak{m}_{X,p}$$
or in English, a germ at $q$ has value zero iff the corresponding germ at $p$ has value zero.

I don't like this formulation $(\pi^\sharp)^{\text{img}}(\mathfrak{m}_{Y,q}) \subseteq \mathfrak{m}_{X,p}$ as much since it hides the geometric intuition behind a lot of symbols: that we want the notion of "value at a point" to be preserved in some way.

At this point, we can state the definition of a scheme, and we do so, although we won't really use it for a few more sections.

**Definition 87.3.3.** A **scheme** is a locally ringed space for which every point has an open neighborhood isomorphic to an affine scheme. A morphism of schemes is just a morphism of locally ringed spaces.

In particular, $\operatorname{Spec} A$ is a scheme (the open neighborhood being the entire space!). And so let's start by looking at those.

## §87.4 A few examples of morphisms between affine schemes

Okay, sorry for lack of examples in previous few sections. Let's make amends now, where you can see all the moving parts in action.

### §87.4.i One-point schemes

**Example 87.4.1** (Spec $\mathbb{R}$ is well-behaved)

There is only one map $X = \operatorname{Spec} \mathbb{R} \to \operatorname{Spec} \mathbb{R} = Y$. Indeed, these are spaces with one point, and specifying the map $\mathbb{R} = \mathcal{O}_Y(Y) \to \mathcal{O}_X(X) = \mathbb{R}$ can only be done in one way, since there is only one field automorphism of $\mathbb{R}$ (the identity).

**Example 87.4.2** (Spec $\mathbb{C}$ horror story)

There are multiple maps $X = \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{C} = Y$, horribly enough! Indeed, these are spaces with one point, so again we're just reduced to specifying a map $\mathbb{C} = \mathcal{O}_Y(Y) \to \mathcal{O}_X(X) = \mathbb{C}$. However, in addition to the identity map, complex conjugation also works, as well as some so-called "wild automorphisms" of $\mathbb{C}$.

This behavior is obviously terrible, so for illustration reasons, some of the examples use $\mathbb{R}$ instead of $\mathbb{C}$ to avoid the horror story we just saw. However, there is an easy fix using "scheme over $\mathbb{C}$" which will force the ring homomorphisms to fix $\mathbb{C}$, later.

**Example 87.4.3** (Spec $k$ and Spec $k'$)

In general, if $k$ and $k'$ are fields, we see that maps $\operatorname{Spec} k \to \operatorname{Spec} k'$ are in bijection with field homomorphism $k' \to k$, since that's all there is left to specify.

### §87.4.ii Examples of constant maps

**Example 87.4.4** (Constant map to $(y-3)$)
We analyze scheme morphisms

$$X = \operatorname{Spec} \mathbb{R}[x] \xrightarrow{\pi} \operatorname{Spec} \mathbb{R}[y] = Y$$

which send all points of $X$ to $\mathfrak{m} = (y-3) \in Y$. Constant maps are continuous no matter how bizarre your topology is, so this lets us just focus our attention on the sections.
This example is simple enough that we can even do it by sections, as much as I think stalks are simpler. Let $U$ be any open subset of $Y$, then we need to specify a map

$$\pi^\sharp_U \colon \mathcal{O}_Y(U) \to \mathcal{O}_X(\pi^{\text{pre}}(U)).$$

If $U$ does not contain $(y-3)$, then $\pi^{\text{pre}}(U) = \varnothing$, so $\mathcal{O}_X(\varnothing) = 0$ is the zero ring and there is nothing to do.
Conversely, if $U$ does contain $(y-3)$ then $\pi^{\text{pre}}(U) = X$, so this time we want to specify a map

$$\pi^\sharp_U \colon \mathcal{O}_Y(U) \to \mathcal{O}_X(X) = \mathbb{R}[x]$$

which satisfies restriction maps. Note that for any $U$, the element $y$ must be mapped to a unit in $\mathbb{R}[x]$; since $1/y$ is a section too for a subset of $U$ not containing $(y)$. In more detail, let $W = U \cap D(y)$ so that $(y) \notin W$, then

$$\pi^\sharp_W(y) = \pi^\sharp_U(y) \quad \text{and} \quad \pi^\sharp_W(y)\pi^\sharp_W(1/y) = 1.$$

Actually for any real number $c \neq 3$, $y - c$ must be mapped to a unit in $\mathbb{R}[x]$. This can only happen if $y \mapsto 3 \in \mathbb{R}[x]$.
As we have specified $\mathbb{R}[y] \mapsto \mathbb{R}[x]$ with $y \mapsto 3$, that determines all the ring homomorphisms we needed.

But we could have used stalks, too. We wanted to specify a morphism

$$\mathbb{R}[y]_{(y-3)} = \mathcal{O}_{\operatorname{Spec} Y,(y-3)} \to \mathcal{O}_{\operatorname{Spec} X,\mathfrak{p}}$$

for every prime ideal $\mathfrak{p}$, sending compatible germs to compatible germs... but wait, $(y-3)$ is spitting out all the germs. So every *individual* germ in $\mathcal{O}_{\operatorname{Spec} Y,(y-3)}$ needs to yield a (compatible) germ above every point of $\operatorname{Spec} X$, which is the data of an entire global section. So we're actually trying to specify

$$\mathbb{R}[y]_{(y-3)} = \mathcal{O}_{\operatorname{Spec} Y,(y-3)} \to \mathcal{O}_{\operatorname{Spec} X}(\operatorname{Spec} X) = \mathbb{R}[x].$$

This requires $y \mapsto 3$, as we saw, since $y - c$ is a unit of $\mathbb{R}[x]$ for any $c \neq 3$.

**Example 87.4.5** (Constant map to $(y^2+1)$ does not exist)
Let's see if there are constant maps $X = \operatorname{Spec} \mathbb{R}[x] \to \operatorname{Spec} \mathbb{R}[y] = Y$ which send everything to $(y^2+1)$. Copying the previous example, we see that we want

$$\mathcal{O}_Y(U) \to \mathcal{O}_X(X) = \mathbb{R}[x].$$

We find that $y$ and $1/y$ have nowhere to go: the same argument as last time shows

that $y - c$ should be a unit of $\mathbb{R}[x]$; this time for any real number $c$.
Like this time, stalks show this too, even with just residue fields. We would for example need a field homomorphism

$$\mathbb{C} = \kappa((y^2+1)) \to \kappa((x)) = \mathbb{R}$$

which does not exist.

You might already notice the following:

**Example 87.4.6** (The generic point repels smaller points)
Changing the tune, consider maps $\operatorname{Spec}\mathbb{C}[x] \to \operatorname{Spec}\mathbb{C}[y]$. We claim that if $\mathfrak{m}$ is a maximal ideal (closed point) of $\mathbb{C}[x]$, then it can never be mapped to the generic point $(0)$ of $\mathbb{C}[y]$.
For otherwise, we would get a local ring homomorphism

$$\mathbb{C}(y) \cong \mathcal{O}_{\operatorname{Spec} k[y],(0)} \to \mathcal{O}_{\operatorname{Spec}\mathbb{C}[x],\mathfrak{m}} \cong \mathbb{C}[x]_{\mathfrak{m}}$$

which in particular means we have a map on the residue fields

$$\mathbb{C}(y) \to \mathbb{C}[x]/\mathfrak{m} \cong \mathbb{C}$$

which is impossible, there is no such field homomorphism at all (why?).

The last example gives some nice intuition in general: "more generic" points tend to have bigger stalks than "less generic" points, hence repel them.

## §87.4.iii The map $t \mapsto t^2$

We now consider what we would think of as the map $t \mapsto t^2$.

**Example 87.4.7** (The map $t \mapsto t^2$)
We consider a map

$$\pi \colon X = \operatorname{Spec}\mathbb{C}[x] \to \operatorname{Spec}\mathbb{C}[y] = Y$$

defined on points as follows:

$$\pi((0)) = (0)$$
$$\pi((x-a)) = (y-a^2).$$

You may check if you wish this map is continuous. I claim that, surprisingly, you can actually read off $\pi^\sharp$ from just this behavior at points. The reason is that we imposed the requirement that a section $s$ can vanish at $\mathfrak{q} \in Y$ if and only if $\pi^\sharp_X(s)$ vanishes at $\mathfrak{p} \in X$, where $\pi(\mathfrak{p}) = \mathfrak{q}$. So, now:

- Consider the section $y \in \mathcal{O}_Y(Y)$, which vanishes only at $(y) \in \operatorname{Spec}\mathbb{C}[y]$; then its image $\pi^\sharp_Y(y) \in \mathcal{O}_X(X)$ must vanish at exactly $(x) \in \operatorname{Spec}\mathbb{C}[x]$, so $\pi^\sharp_Y(y) = x^n$ for some integer $n \geq 1$.

- Consider the section $y-4 \in \mathcal{O}_Y(Y)$, which vanishes only at $(y-4) \in \operatorname{Spec} \mathbb{C}[y]$; then its image $\pi_Y^\sharp(y-4) \in \mathcal{O}_X(X)$ must vanish at exactly $(x-2) \in \operatorname{Spec} \mathbb{C}[x]$ and $(x+2) \in \operatorname{Spec} \mathbb{C}[x]$. So $\pi_Y^\sharp(y) - 4$ is divisible by $(x-2)^a(x+2)^b$ for some $a \geq 1$ and $b \geq 1$.

Thus $y \mapsto x^2$ in the top level map of sections $\pi_Y^\sharp$: and hence also in all the maps of sections (as well as at all the stalks).

The above example works equally well if $t^2$ is replaced by some polynomial $f(t)$, so that $(x-a)$ maps to $(y - f(y))$. The image of $y$ must be a polynomial $g(x)$ with the property that $g(x) - c$ has the same roots as $f(x) - c$ for any $c \in \mathbb{C}$. Put another way, $f$ and $g$ have the same values, so $f = g$.

**Remark 87.4.8** (Generic point stalk overpowered) — I want to also point out that you can read off the polynomial just from the stalk at the generic point: for example, the previous example has

$$\mathbb{C}(y) \cong \mathcal{O}_{\operatorname{Spec} \mathbb{C}[y],(0)} \to \mathcal{O}_{\operatorname{Spec} \mathbb{C}[x],(0)} \cong \mathbb{C}(x) \qquad y \mapsto x^2.$$

This is part of the reason why generic points are so powerful. We expect that with polynomials, if you know what happens to a "generic" point, you can figure out the entire map. This intuition is true: knowing where each germ at the generic point goes is enough to tell us the whole map.

### §87.4.iv An arithmetic example

**Example 87.4.9** ($\operatorname{Spec} \mathbb{Z}[i] \to \operatorname{Spec} \mathbb{Z}$)

We now construct a morphism of schemes $\pi\colon \operatorname{Spec} \mathbb{Z}[i] \to \operatorname{Spec} \mathbb{Z}$. On points it behaves by

$$\begin{aligned} \pi\left((0)\right) &= (0) \\ \pi\left((p)\right) &= (p) \\ \pi\left((a+bi)\right) &= (a^2+b^2) \end{aligned}$$

where $a+bi$ is a Gaussian prime: so for example $\pi((2+i)) = (5)$ and $\pi((1+i)) = (2)$. We could figure out the induced map on stalks now, much like before, but in a moment we'll have a big theorem that spares us the trouble.

## §87.5 The big theorem

We did a few examples of $\operatorname{Spec} A \to \operatorname{Spec} B$ by hand, specifying the full data of a map of locally ringed spaces. It turns out that in fact, we didn't to specify that much data, and much of the process can be automated:

**Proposition 87.5.1** (Affine reconstruction)

Let $\pi \colon \operatorname{Spec} A \to \operatorname{Spec} B$ be a map of schemes. Let $\psi \colon B \to A$ be the ring homomorphism obtained by taking global sections, i.e.

$$\psi = \pi^\sharp_B \colon \mathcal{O}_{\operatorname{Spec} B}(\operatorname{Spec} B) \to \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A).$$

Then we can recover $\pi$ given only $\psi$; in fact, $\pi$ is given explicitly by

$$\pi(\mathfrak{p}) = \psi^{\text{pre}}(\mathfrak{p})$$

and

$$\pi^\sharp_{\mathfrak{p}} \colon \mathcal{O}_{Y,\pi(\mathfrak{p})} \to \mathcal{O}_{X,\mathfrak{p}} \quad \text{by} \quad f/g \mapsto \psi(f)/\psi(g).$$

This is the big miracle of affine schemes. Despite the enormous amount of data packaged into the definition, we can compress maps between affine schemes into just the single ring homomorphism on the top level.

*Proof.* This requires two parts.

- We need to check that the maps agree on *points*; surprisingly this is the harder half. To see how this works, let $\mathfrak{q} = \pi(\mathfrak{p})$. The key fact is that a function $f \in B$ vanishes on $\mathfrak{q}$ if and only if $\pi^\sharp_B(f)$ vanishes on $\mathfrak{p}$ (because $\pi^\sharp_B$ is supposed to be a homomorphism of *local* rings). Therefore,

$$\begin{aligned}
\pi(\mathfrak{p}) = \mathfrak{q} &= \{f \in B \mid f \in \mathfrak{q}\} \\
&= \{f \in B \mid f \text{ vanishes on } \mathfrak{q}\} \\
&= \left\{f \in B \mid \pi^\sharp_B(f) \text{ vanishes on } \mathfrak{p}\right\} \\
&= \left\{f \in B \mid \pi^\sharp_B(f) \in \mathfrak{p}\right\} = \{f \in B \mid \psi(f) \in \mathfrak{p}\} \\
&= \psi^{\text{pre}}(\mathfrak{p}).
\end{aligned}$$

- We also want to check the maps on the stalks is the same. Suppose $\mathfrak{p} \in \operatorname{Spec} A$, $\mathfrak{q} \in \operatorname{Spec} B$, and $\mathfrak{p} \mapsto \mathfrak{q}$ (under both of the above).

  In our original $\pi$, consider the map $\pi^\sharp_{\mathfrak{p}} \colon B_{\mathfrak{q}} \to A_{\mathfrak{p}}$. We know that it sends each $f \in B$ to $\psi(f) \in A$, by taking the germ of each global section $f \in B$ at $\mathfrak{q}$. Thus it must send $f/g$ to $\psi(f)/\psi(g)$, being a ring homomorphism, as needed. $\square$

All of this suggests a great idea: if $\psi \colon B \to A$ is *any* ring homomorphism, we ought to be able to construct a map of schemes by using fragments of the proof we just found. The only extra work we have to do is verify that we get a continuous map in the Zariski topology, and that we can get a suitable $\pi^\sharp$.

We thus get the huge important theorem about affine schemes.

**Theorem 87.5.2** ($\operatorname{Spec} A \to \operatorname{Spec} B$ is just $B \to A$)

These two construction gives a bijection between ring homomorphisms $B \to A$ and $\operatorname{Spec} A \to \operatorname{Spec} B$.

*Proof.* We have seen how to take each $\pi \colon \operatorname{Spec} A \to \operatorname{Spec} B$ and get a ring homomorphism $\psi$. Proposition 87.5.1 shows this map is injective. So we just need to check it is surjective — that every $\psi$ arises from some $\pi$.

Given $\psi \colon B \to A$, we define $(\pi, \pi^\sharp) \colon \operatorname{Spec} A \to \operatorname{Spec} B$ by copying Proposition 87.5.1 and checking that everything is well-definced. The details are:

- For each prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, we let $\pi(\mathfrak{p}) = \psi^{\text{pre}}(\mathfrak{p}) \in \operatorname{Spec} B$ (which by Problem 5C$^\star$ is also prime).

> **Exercise 87.5.3.** Show that the resulting map $\pi$ is continuous in the Zariski topology.

- Now we want to also define maps on the stalks, and so for each $\pi(\mathfrak{p}) = \mathfrak{q}$ we set
$$B_\mathfrak{q} \ni \frac{f}{g} \mapsto \frac{\psi(f)}{\psi(g)} \in A_\mathfrak{p}.$$
This makes sense since $g \notin \mathfrak{q} \implies \psi(g) \notin \mathfrak{p}$. Also $f \in \mathfrak{q} \implies \psi(f) \in \mathfrak{p}$, we find this really is a local ring homomorphism (sending the maximal ideal of $B_\mathfrak{q}$ into the one of $A_\mathfrak{p}$).

  Observe that if $f/g$ is a *section* over an open set $U \subseteq B$ (meaning $g$ does not vanish at the primes in $U$), then $\psi(f)/\psi(g)$ is a section over $\pi^{\text{pre}}(U)$ (meaning $\psi(g)$ does not vanish at the primes in $\pi^{\text{pre}}(U)$). Therefore, compatible germs over $B$ get sent to compatible germs over $A$, as needed.

Finally, the resulting $\pi$ has $\pi^\sharp_B = \psi$ on global sections, completing the proof. $\square$

This can be summarized succinctly using category theory:

**Corollary 87.5.4** (Categorical interpretation)
The opposite category of rings $\mathsf{CRing}^{\text{op}}$, is "equivalent" to the category of affine schemes, $\mathsf{AffSch}$, with Spec as a functor.

This means for example that $\operatorname{Spec} A \cong \operatorname{Spec} B$, naturally, whenever $A \cong B$.

To make sure you realize that this theorem is important, here is an amusing comment I found on MathOverflow while reading about algebraic geometry references[3]:

> *He [Hartshorne] never mentions that the category of affine schemes is dual to the category of rings, as far as I can see. I'd expect to see that in huge letters near the definition of scheme. How could you miss that out?*

## §87.6 More examples of scheme morphisms

Now that we have the big hammer, we can talk about examples much more briefly than we did a few sections ago. Before throwing things around, I want to give another definition that will eliminate the weird behavior we saw with $\mathbb{C} \to \mathbb{C}$ having nontrivial field automorphisms:

**Definition 87.6.1.** Let $S$ be a scheme. A **scheme over $S$** or **$S$-scheme** is a scheme $X$ together with a map $X \to S$. A morphism of $S$-schemes is a scheme morphism $X \to Y$ such that the diagram
$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}$$
commutes. Often, if $S = \operatorname{Spec} k$, we will refer to $X$ by schemes over $k$ or $k$-schemes for short.

[3]From https://mathoverflow.net/q/2446/70654.

**Example 87.6.2** (Spec $k[\dots]$)

If $X = \operatorname{Spec} k[x_1, \dots, x_n]/I$ for some ideal $I$, then $X$ is a $k$-scheme in a natural way; since we have an obvious homomorphism $k \hookrightarrow k[x_1, \dots, x_n]/I$ which gives a map $X \to \operatorname{Spec} k$.

**Example 87.6.3** ($\operatorname{Spec} \mathbb{C}[x] \to \operatorname{Spec} \mathbb{C}[y]$)

As $\mathbb{C}$-schemes, maps $\operatorname{Spec} \mathbb{C}[x] \to \operatorname{Spec} \mathbb{C}[y]$ coincide with ring homomorphisms from $\psi\colon \mathbb{C}[y] \to \mathbb{C}[x]$ such that the diagram

$$\begin{tikzcd}
\mathbb{C}[x] & \mathbb{C}[y] \arrow[l, "\psi"'] \\
& \mathbb{C} \arrow[ul, hook] \arrow[u, hook]
\end{tikzcd}$$

commutes. We see that the "over $\mathbb{C}$" condition is eliminating the pathology from before: the $\psi$ is required to preserve $\mathbb{C}$. So the morphism is determined by the image of $y$, i.e. the choice of a polynomial in $\mathbb{C}[x]$. For example, if $\psi(y) = x^2$ we recover the first example we saw. This matches our intuition that these maps should correspond to polynomials.

**Example 87.6.4** ($\operatorname{Spec} \mathcal{O}_K$)

This generalizes $\operatorname{Spec} \mathbb{Z}[i]$ from before. If $K$ is a number field and $\mathcal{O}_K$ is the ring of integers, then there is a natural morphism $\operatorname{Spec} \mathcal{O}_K \to \operatorname{Spec} \mathbb{Z}$ from the (unique) ring homomorphism $\mathbb{Z} \hookrightarrow \mathcal{O}_K$. Above each rational prime $(p) \in \mathbb{Z}$, one obtains the prime ideals that $p$ splits as. (We don't have a way of capturing ramification yet, alas.)

## §87.7 A little bit on non-affine schemes

We can finally state the isomorphism that we wanted for a long time (first mentioned in Section 85.2.iii):

**Theorem 87.7.1** (Distinguished open sets are isomorphic to affine schemes)

Let $A$ be a ring and $f$ an element. Then

$$\operatorname{Spec} A[1/f] \cong D(f) \subseteq \operatorname{Spec} A.$$

*Proof.* Annoying check, not included yet. (We have already seen the bijection of prime ideals, at the level of points.) $\square$

**Corollary 87.7.2** (Open subsets are schemes)

(a) Any nonempty open subset of an affine scheme is itself a scheme.

(b) Any nonempty open subset of any scheme (affine or not) is itself a scheme.

*Proof.* Part (a) has essentially been done already:

> **Question 87.7.3.** Combine Theorem 85.2.2 with the previous proposition to deduce (a).

Part (b) then follows by noting that if $U$ is an open set, and $p$ is a point in $U$, then we can take an affine open neighborhood $\operatorname{Spec} A$ at $p$, and then cover $U \cap \operatorname{Spec} A$ with distinguished open subsets of $\operatorname{Spec} A$ as in (a). $\square$

We now reprise Section 85.2.iv (except $\mathbb{C}$ will be replaced by $k$). We have seen it is an open subset $U$ of $\operatorname{Spec} k[x, y]$, so it is a scheme.

> **Question 87.7.4.** Show that in fact $U$ can be covered by two open sets which are both affine.

However, we show now that you really do need two distinguished open sets.

**Proposition 87.7.5** (Famous example: punctured plane isn't affine)

The punctured plane $U = (U, \mathcal{O}_U)$, obtained by deleting $(x, y)$ from $\operatorname{Spec} k[x, y]$, is not isomorphic to any affine scheme $\operatorname{Spec} B$.

The intuition is that $\mathcal{O}_U(U) = k[x, y]$, but $U$ is not the plane.

*Proof.* We already know $\mathcal{O}_U(U) = k[x, y]$ and we have a good handle on it. For example, $y \in \mathcal{O}_U(U)$ is a global section which vanishes on what looks like the $y$-axis. Similarly, $x \in \mathcal{O}_X(X)$ is a global section which vanishes on what looks like the $y$-axis. In particular, no point of $U$ vanishes at both.

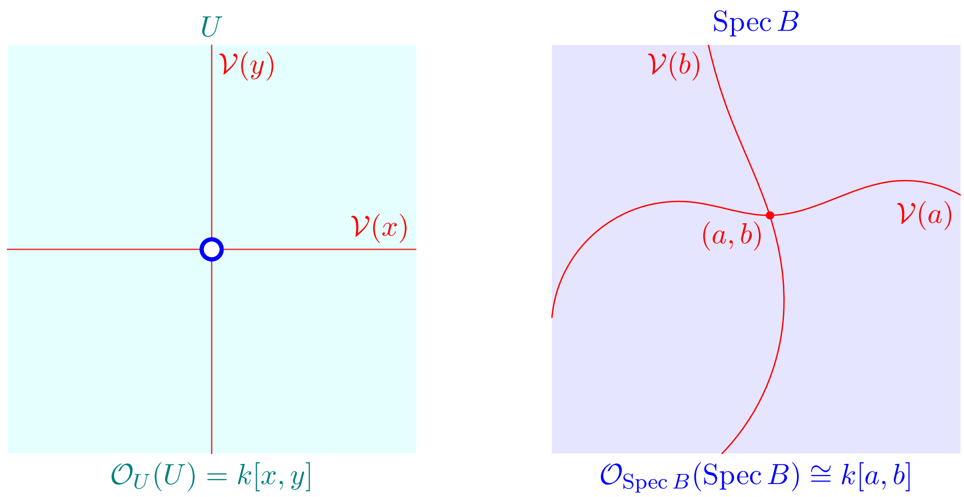
Now assume for contradiction that we have an isomorphism

$$\psi \colon \operatorname{Spec} B \to U.$$

By taking the map on global sections (part of the definition),

$$k[x, y] = \mathcal{O}_U(U) \xrightarrow{\psi^\sharp} \mathcal{O}_{\operatorname{Spec} B}(\operatorname{Spec} B) \cong B.$$

The global sections $x$ and $y$ in $\mathcal{O}_U(U)$ should then have images $a$ and $b$ in $B$; and it follows we have a ring isomorphism $B \cong k[a, b]$.



Now in $\operatorname{Spec} B$, $\mathcal{V}(a) \cap \mathcal{V}(b)$ is a closed set containing a single point, the maximal ideal $\mathfrak{m} = (a, b)$. Thus in $\operatorname{Spec} B$ there is exactly one point vanishing at both $a$ and $b$. *Because we required morphisms of schemes to preserve values* (hence the big fuss about locally ringed spaces), that means there should be a single point of $U$ vanishing at both $x$ and $y$. But there isn't — it was the origin we deleted. $\square$

## §87.8 Where to go from here

This chapter concludes the long setup for the definition of a scheme. For now, this unfortunately is as far as I have time to go. So, if you want to actually see how schemes are used in "real life", you'll have to turn elsewhere.

A good reference I happily recommend is [**Ga03**]; a more difficult (and famous) one is [**Va17**]. See Appendix A for further remarks.

## §87.9 A few harder problems to think about

**Problem 87A.** Given an affine scheme $X = \operatorname{Spec} R$, show that there is a unique morphism of schemes $X \to \operatorname{Spec} \mathbb{Z}$, and describe where it sends points of $X$.

**Problem 87B.** Is the open subset of $\operatorname{Spec} \mathbb{Z}[x]$ obtained by deleting the point $\mathfrak{m} = (2, x)$ isomorphic to some affine scheme?