XIX

Algebraic Geometry I: Classical Varieties

Part XIX: Contents

77 Affine varieties

In this chapter we introduce affine varieties. We introduce them in the context of coordinates, but over the course of the other chapters we'll gradually move away from this perspective to viewing varieties as "intrinsic objects", rather than embedded in coordinates.

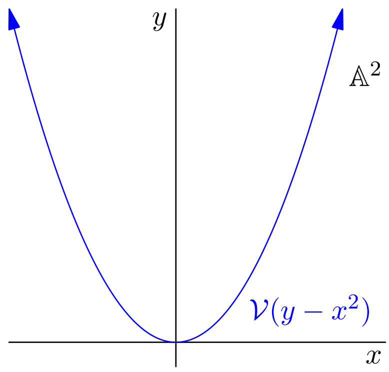
For simplicity, we'll do almost everything over the field of complex numbers, but the discussion generalizes to any algebraically closed field.

§77.1 Affine varieties

Prototypical example for this section: $\mathcal{V}(y - x^2)$ *is a parabola in* $\mathbb{A}^2$.

Definition 77.1.1. Given a set of polynomials $S \subseteq \mathbb{C}[x_1, \dots, x_n]$ (not necessarily finite or even countable), we let $\mathcal{V}(S)$ denote the set of points vanishing on *all* the polynomials in S. Such a set is called an **affine variety**. It lives in **n-dimensional affine space**, denoted $\mathbb{A}^n$ (to distinguish it from projective space later).

For example, a parabola is the zero locus of the polynomial $y - x^2$. Picture:



Example 77.1.2 (Examples of affine varieties)

These examples are in two-dimensional space $\mathbb{A}^2$, whose points are pairs (x, y).

(a) A straight line can be thought of as $\mathcal{V}(Ax + By + C)$.

(b) A parabola as above can be pictured as $\mathcal{V}(y - x^2)$.

(c) A hyperbola might be $\mathcal{V}(xy - 1)$.

(d) The two axes can be thought of as $\mathcal{V}(xy)$; this is the set of points such that $x = 0$ *or* $y = 0$.

(e) A point (x_0, y_0) can be thought of as $\mathcal{V}(x - x_0, y - y_0)$.

(f) The entire space $\mathbb{A}^2$ can be thought of as $\mathcal{V}(0)$.

(g) The empty set is the zero locus of the constant polynomial 1, that is $\mathcal{V}(1)$.

§77.2 Naming affine varieties via ideals

Prototypical example for this section: $\mathcal{V}(I)$ *is a parabola, where* $I = (y - x^2)$.

As you might have already noticed, a variety can be named by $\mathcal{V}(-)$ in multiple ways. For example, the set of solutions to

$$x = 3 \text{ and } y = 4$$

is just the point $(3,4)$. But this is also the set of solutions to

$$x = 3 \text{ and } y = x + 1.$$

So, for example

$$\{(3,4)\} = \mathcal{V}(x-3, y-4) = \mathcal{V}(x-3, y-x-1).$$

That's a little annoying, because in an ideal[1] world we would have *one* name for every variety. Let's see if we can achieve this.

A partial solution is to use *ideals* rather than small sets. That is, consider the ideal

$$I = (x-3, y-4) = \{p(x,y) \cdot (x-3) + q(x,y) \cdot (y-4) \mid p, q \in \mathbb{C}[x,y]\}$$

and look at $\mathcal{V}(I)$.

> **Question 77.2.1.** Convince yourself that $\mathcal{V}(I) = \{(3,4)\}$.

So rather than writing $\mathcal{V}(x-3, y-4)$ it makes sense to think about this as $\mathcal{V}(I)$, where $I = (x-3, y-4)$ is the *ideal* generated by the two polynomials $x-3$ and $y-4$. This is an improvement because

> **Question 77.2.2.** Check that $(x-3, y-x-1) = (x-3, y-4)$.

Needless to say, this pattern holds in general.

> **Question 77.2.3.** Let $\{f_i\}$ be a set of polynomials, and consider the ideal I generated by these $\{f_i\}$. Show that $\mathcal{V}(\{f_i\}) = \mathcal{V}(I)$.

Thus we will only consider $\mathcal{V}(I)$ when I is an ideal. Of course, frequently our ideals are generated by one or two polynomials, which leads to:

Abuse of Notation 77.2.4. Given a set of polynomials $f_1, \dots, f_m$ we let $\mathcal{V}(f_1, \dots, f_m)$ be shorthand for $\mathcal{V}((f_1, \dots, f_m))$. In other words we let $\mathcal{V}(f_1, \dots, f_m)$ abbreviate $\mathcal{V}(I)$, where I is the *ideal* $I = (f_1, \dots, f_m)$.

This is where the Noetherian condition really shines: it guarantees that every ideal $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ can be written in the form above with *finitely* many polynomials, because it is *finitely generated*. (The fact that $\mathbb{C}[x_1, \dots, x_n]$ is Noetherian follows from the Hilbert basis theorem, which is Theorem 4.9.5). This is a relief, because dealing with infinite sets of polynomials is not much fun.

[1] Pun not intended but left for amusement value.

§77.3 Radical ideals and Hilbert's Nullstellensatz

Prototypical example for this section: $\sqrt{(x^2)} = (x)$ *in* $\mathbb{C}[x]$, $\sqrt{(12)} = (6)$ *in* $\mathbb{Z}$.

You might ask whether the name is unique now: that is, if $\mathcal{V}(I) = \mathcal{V}(J)$, does it follow that $I = J$? The answer is unfortunately no: the counterexample can be found in just $\mathbb{A}^1$. It is

$$\mathcal{V}(x) = \mathcal{V}(x^2).$$

In other words, the set of solutions to $x = 0$ is the same as the set of solutions to $x^2 = 0$.

Well, that's stupid. We want an operation which takes the ideal (x^2) and makes it into the ideal (x). The way to do so is using the radical of an ideal.

Definition 77.3.1. Let R be a ring. The **radical** of an ideal $I \subseteq R$, denoted $\sqrt{I}$, is defined by

$$\sqrt{I} = \{r \in R \mid r^m \in I \text{ for some integer } m \geq 1\}.$$

If $I = \sqrt{I}$, we say the ideal I itself is **radical**.

For example, $\sqrt{(x^2)} = (x)$. You may like to take the time to verify that $\sqrt{I}$ is actually an ideal.

Remark 77.3.2 (Number theoretic motivation) — This is actually the same as the notion of "radical" in number theory. In $\mathbb{Z}$, the radical of an ideal (n) corresponds to just removing all the duplicate prime factors, so for example

$$\sqrt{(12)} = (6).$$

In particular, if you try to take $\sqrt{(6)}$, you just get (6) back; you don't squeeze out any new prime factors.

This is actually true more generally, and there is a nice corresponding alternate definition: for any ideal I, we have

$$\sqrt{I} = \bigcap_{I \subseteq \mathfrak{p} \text{ prime}} \mathfrak{p}.$$

Although we could prove this now, it will be proved later in Theorem 84.4.2, when we first need it.

Here are the immediate properties you should know.

Proposition 77.3.3 (Properties of radical)

In any ring:

- If I is an ideal, then $\sqrt{I}$ is always a radical ideal.
- Prime ideals are radical.
- For $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ we have $\mathcal{V}(I) = \mathcal{V}(\sqrt{I})$.

Proof. These are all obvious.

- If $f^m \in \sqrt{I}$ then $f^{mn} \in I$, so $f \in \sqrt{I}$.

- If $f^n \in \mathfrak{p}$ for a prime $\mathfrak{p}$, then either $f \in \mathfrak{p}$ or $f^{n-1} \in \mathfrak{p}$, and in the latter case we may continue by induction.
- We have $f(x_1, \dots, x_n) = 0$ if and only if $f(x_1, \dots, x_n)^m = 0$ for some integer m. $\square$

The last bit makes sense: you would never refer to $x = 0$ as $x^2 = 0$, and hence we would always want to call $\mathcal{V}(x^2)$ just $\mathcal{V}(x)$. With this, we obtain a theorem called Hilbert's Nullstellensatz.

Theorem 77.3.4 (Hilbert's Nullstellensatz)
Given an affine variety $V = \mathcal{V}(I)$, the set of polynomials which vanish on all points of V is precisely $\sqrt{I}$. Thus if I and J are ideals in $\mathbb{C}[x_1, \dots, x_n]$, then

$$\mathcal{V}(I) = \mathcal{V}(J) \text{ if and only if } \sqrt{I} = \sqrt{J}.$$

In other words

Radical ideals in $\mathbb{C}[x_1, \dots, x_n]$ correspond exactly to n-dimensional affine varieties.

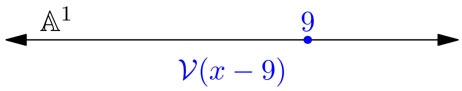
The proof of Hilbert's Nullstellensatz will be given in Problem 77D; for now it is worth remarking that it relies essentially on the fact that $\mathbb{C}$ is *algebraically closed*. For example, it is false in $\mathbb{R}[x]$, with $(x^2 + 1)$ being a maximal ideal with empty vanishing set.

§77.4 Pictures of varieties in $\mathbb{A}^1$

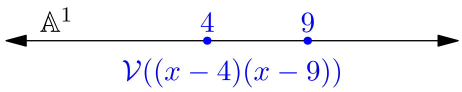
Prototypical example for this section: Finite sets of points (in fact these are the only nontrivial examples).

Let's first draw some pictures. In what follows I'll draw $\mathbb{C}$ as a straight line... sorry.

First of all, let's look at just the complex line $\mathbb{A}^1$. What are the various varieties on it? For starters, we have a single point $9 \in \mathbb{C}$, generated by $(x - 9)$.



Another example is the point 4. And in fact, if we like we can get an ideal consisting of just these two points; consider $\mathcal{V}((x - 4)(x - 9))$.



In general, in $\mathbb{A}^1$ you can get finitely many points $\{a_1, \dots, a_n\}$ by just taking

$$\mathcal{V}((x - a_1)(x - a_2) \dots (x - a_n)).$$

On the other hand, you can't get the set $\{0, 1, 2, \dots\}$ as an affine variety; the only polynomial vanishing on all those points is the zero polynomial. In fact, you can convince yourself that these are the only affine varieties, with two exceptions:

- The entire line $\mathbb{A}^1$ is given by $\mathcal{V}(0)$, and
- The empty set is given by $\mathcal{V}(1)$.

Exercise 77.4.1. Show that these are the only varieties of $\mathbb{A}^1$. (Let $\mathcal{V}(I)$ be the variety and pick a $0 \neq f \in I$.)

As you might correctly guess, we have:

Theorem 77.4.2 (Intersections and unions of varieties)

(a) The intersection of affine varieties (even infinitely many) is an affine variety.

(b) The union of finitely many affine varieties is an affine variety.

In fact we have

$$\bigcap_\alpha \mathcal{V}(I_\alpha) = \mathcal{V}\left(\sum_\alpha I_\alpha\right) \qquad \text{and} \qquad \bigcup_{k=1}^n \mathcal{V}(I_k) = \mathcal{V}\left(\bigcap_{k=1}^n I_k\right).$$

You are welcome to prove this easy result yourself.

Remark 77.4.3 — Part (a) is a little misleading in that the sum $I + J$ need not be radical: take for example $I = (y - x^2)$ and $J = (y)$ in $\mathbb{C}[x, y]$, where $x \in \sqrt{I + J}$ and $x \notin I + J$. But in part (b) for radical ideals I and J, the intersection $I \cap J$ is radical.

As another easy result concerning the relation between the ideal and variety, we have:

Proposition 77.4.4 ($\mathcal{V}(-)$ is inclusion reversing)

If $I \subseteq J$ then $\mathcal{V}(I) \supseteq \mathcal{V}(J)$. Thus $\mathcal{V}(-)$ is *inclusion-reversing*.

Question 77.4.5. Verify this.

Thus, bigger ideals correspond to smaller varieties.

These results will be used a lot throughout the chapter, so it would be useful for you to be comfortable with the inclusion-reversing nature of $\mathcal{V}$.

Exercise 77.4.6. Some quick exercises to help you be more familiar with the concepts.

1. Let $I = (y - x^2)$ and $J = (x + 1, y + 2)$. What is $\mathcal{V}(I)$ and $\mathcal{V}(J)$?
2. What is the ideal K such that $\mathcal{V}(K)$ is the union of the parabola $y = x^2$ and the point $(-1, -2)$?
3. Let $L = (y - 1)$. What is $\mathcal{V}(L)$?
4. The intersection $\mathcal{V}(I) \cap \mathcal{V}(L)$ consist of two points $(1, 1)$ and $(-1, 1)$. What's the ideal corresponding to it, in terms of I and L?
5. What is $\mathcal{V}(I \cap L)$? What about $\mathcal{V}(IL)$?

Question 77.4.7. Note that the intersection of infinitely many ideals is still an ideal, but the union of infinitely many affine varieties may not be an affine variety.

> Consider $I_k = (x-k)$ in $\mathbb{C}[x]$, and take the infinite intersection $I = \bigcap_{k\in\mathbb{N}} I_k$. What is $\mathcal{V}(I)$ and $\bigcup_{k\in\mathbb{N}} \mathcal{V}(I_k)$?

§77.5 Prime ideals correspond to irreducible affine varieties

Prototypical example for this section: (xy) corresponds to the union of two lines in $\mathbb{A}^2$.

Note that most of the affine varieties of $\mathbb{A}^1$, like $\{4,9\}$, are just unions of the simplest "one-point" ideals. To ease our classification, we can restrict our attention to the case of *irreducible* varieties:

Definition 77.5.1. A variety V is **irreducible** if it cannot be written as the union of two proper sub-varieties $V = V_1 \cup V_2$.

Abuse of Notation 77.5.2. Warning: in other literature, irreducible is part of the definition of variety.

Example 77.5.3 (Irreducible varieties of $\mathbb{A}^1$)
The irreducible varieties of $\mathbb{A}^1$ are:

- the empty set $\mathcal{V}(1)$,
- a single point $\mathcal{V}(x-a)$, and
- the entire line $\mathbb{A}^1 = \mathcal{V}(0)$.

Example 77.5.4 (The union of two axes)
Let's take a non-prime ideal in $\mathbb{C}[x,y]$, such as $I = (xy)$. Its vanishing set $\mathcal{V}(I)$ is the union of two lines $x=0$ and $y=0$. So $\mathcal{V}(I)$ is reducible.

In general:

Theorem 77.5.5 (Prime $\iff$ irreducible)
Let I be a radical ideal, and $V = \mathcal{V}(I)$ a nonempty variety. Then I is prime if and only if V is irreducible.

Proof. First, assume V is irreducible; we'll show I is prime. Let $f,g \in \mathbb{C}[x_1,\dots,x_n]$ so that $fg \in I$. Then V is a subset of the union $\mathcal{V}(f) \cup \mathcal{V}(g)$; actually, $V = (V \cap \mathcal{V}(f)) \cup (V \cap \mathcal{V}(g))$. Since V is irreducible, we may assume $V = V \cap \mathcal{V}(f)$, hence f vanishes on all of V. So $f \in I$.

The reverse direction is similar. $\square$

§77.6 Pictures in $\mathbb{A}^2$ and $\mathbb{A}^3$

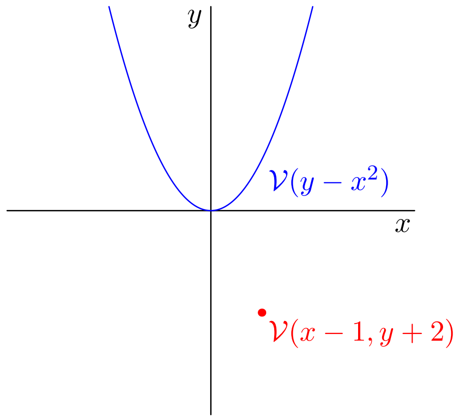
Prototypical example for this section: Various curves and hypersurfaces.

With this notion, we can now draw pictures in "complex affine plane", $\mathbb{A}^2$. What are the irreducible affine varieties in it?

As we saw in the previous discussion, naming irreducible affine varieties in $\mathbb{A}^2$ amounts to naming the prime ideals of $\mathbb{C}[x,y]$. Here are a few.

- The ideal (0) is prime. $\mathcal{V}(0)$ as usual corresponds to the entire plane.
- The ideal $(x-a, y-b)$ is prime, since $\mathbb{C}[x,y]/(x-a,y-b) \cong \mathbb{C}$ is an integral domain. (In fact, since $\mathbb{C}$ is a field, the ideal $(x-a,y-b)$ is *maximal*). The vanishing set of this is $\mathcal{V}(x-a,y-b) = \{(a,b)\} \in \mathbb{C}^2$, so these ideals correspond to a single point.
- Let $f(x,y)$ be an irreducible polynomial, like $y - x^2$. Then (f) is a prime ideal! Here $\mathcal{V}(I)$ is a "degree one curve".

By using some polynomial algebra (again you're welcome to check this; Euclidean algorithm), these are in fact the only prime ideals of $\mathbb{C}[x,y]$. Here's a picture.



As usual, you can make varieties which are just unions of these irreducible ones. For example, if you wanted the variety consisting of a parabola $y = x^2$ plus the point $(20,15)$ you would write

$$\mathcal{V}\left((y-x^2)(x-20), (y-x^2)(y-15)\right).$$

The picture in $\mathbb{A}^3$ is harder to describe. Again, you have points $\mathcal{V}(x-a,y-b,z-c)$ corresponding to be zero-dimensional points (a,b,c), and two-dimensional surfaces $\mathcal{V}(f)$ for each irreducible polynomial f (for example, $x+y+z=0$ is a plane). But there are more prime ideals, like $\mathcal{V}(x,y)$, which corresponds to the intersection of the planes $x=0$ and $y=0$: this is the one-dimensional z-axis. It turns out there is no reasonable way to classify the "one-dimensional" varieties; they correspond to "irreducible curves".

Thus, as Ravi Vakil [Va17] says: the purely algebraic question of determining the prime ideals of $\mathbb{C}[x,y,z]$ has a fundamentally geometric answer.

§77.7 Maximal ideals

Prototypical example for this section: All maximal ideals are $(x_1-a_1,\dots,x_n-a_n)$.

Recall that bigger ideals correspond to smaller varieties.

As the above pictures might have indicated, the smallest varieties are *single points*. Moreover, as you might guess from the name, the biggest ideals are the *maximal ideals*. As an example, all ideals of the form

$$(x_1-a_1,\dots,x_n-a_n)$$

are maximal, since the quotient

$$\mathbb{C}[x_1,\dots,x_n]/(x_1-a_1,\dots,x_n-a_n) \cong \mathbb{C}$$

is a field. The question is: are all maximal ideals of this form?

The answer is in the affirmative.

Theorem 77.7.1 (Weak Nullstellensatz, phrased with maximal ideals)
Every maximal ideal of $\mathbb{C}[x_1, \dots, x_n]$ is of the form $(x_1 - a_1, \dots, x_n - a_n)$.

The proof of this is surprisingly nontrivial, so we won't include it here yet; see [**Va17**, §7.4.3]. Again this uses the fact that $\mathbb{C}$ is algebraically closed. (For example (x^2+1) is a maximal ideal of $\mathbb{R}[x]$.) Thus:

Over $\mathbb{C}$, maximal ideals correspond to single points.

Consequently, our various ideals over $\mathbb{C}$ correspond to various flavors of affine varieties:

Algebraic flavor	Geometric flavor
radical ideal	affine variety
prime ideal	irreducible variety
maximal ideal	single point
any ideal	(scheme?)

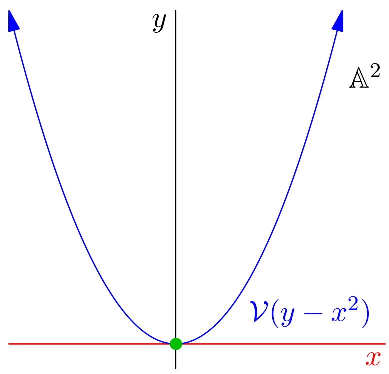
There's one thing I haven't talked about: what's the last entry?

§77.8 Motivating schemes with non-radical ideals

One of the most elementary motivations for the scheme is that we would like to use them to count multiplicity. That is, consider the intersection

$$\mathcal{V}(y - x^2) \cap \mathcal{V}(y) \subseteq \mathbb{A}^2$$

This is the intersection of the parabola with the tangent x-axis, this is the green dot below.



Unfortunately, as a variety, it is just a single point! However, we want to think of this as a "double point": after all, in some sense it has multiplicity 2. You can detect this when you look at the ideals:

$$(y - x^2) + (y) = (x^2, y)$$

and thus, if we blithely ignore taking the radical, we get

$$\mathbb{C}[x, y]/(x^2, y) \cong \mathbb{C}[\varepsilon]/(\varepsilon^2).$$

So the ideals in question are noticing the presence of a double point.

In order to encapsulate this, we need a more refined object than a variety, which (at the end of the day) is just a set of points; it's not possible using topology along to encode more information (there is only one topology on a single point!). This refined object is the *scheme*.

§77.9 A few harder problems to think about

some actual computation here would be good

Problem 77A. Show that $I \subseteq \sqrt{I}$ and $\sqrt{I} \cap \sqrt{J} = \sqrt{IJ} \subseteq \sqrt{I \cap J}$, for two ideals I and J.

Problem 77B. Show that a *real* affine variety $V \subseteq \mathbb{A}^n_{\mathbb{R}}$ can always be written in the form $\mathcal{V}(f)$.

Problem 77C (Complex varieties can't be empty)**.** Prove that if I is a proper ideal in $\mathbb{C}[x_1, \dots, x_n]$ then $\mathcal{V}(I) \neq \varnothing$.

Problem 77D. Show that Hilbert's Nullstellensatz in n dimensions follows from the Weak Nullstellensatz. (This solution is called the **Rabinowitsch Trick**.)

78 Affine varieties as ringed spaces

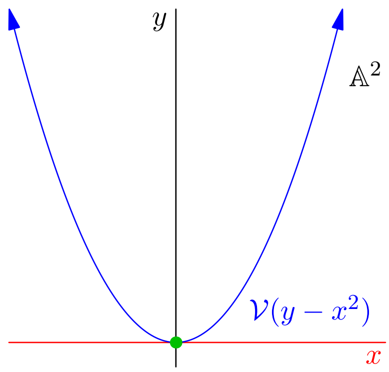
As in the previous chapter, we are working only over affine varieties in $\mathbb{C}$ for simplicity.

§78.1 Synopsis

Group theory was a strange creature in the early 19th century. During the 19th century, a group was literally defined as a subset of $\mathrm{GL}(n)$ or of S_n. Indeed, the word "group" hadn't been invented yet. This may sound ludicrous, but it was true – Sylow developed his theorems without this notion. Only much later was the abstract definition of a group given, an abstract set G which was *independent* of any embedding into S_n, and an object in its own right.

We are about to make the same type of change for our affine varieties. Rather than thinking of them as an object locked into an ambient space $\mathbb{A}^n$ we are instead going to try to make them into an object in their own right. Specifically, for us an affine variety will become a *topological space* equipped with a *ring of functions* for each of its open sets: this is why we call it a **ringed space**.

The bit about the topological space is not too drastic. The key insight is the addition of the ring of functions. For example, consider the double point from last chapter.



As a set, it is a single point, and thus it can have only one possible topology. But the addition of the function ring will let us tell it apart from just a single point.

This construction is quite involved, so we'll proceed as follows: we'll define the structure bit by bit onto our existing affine varieties in $\mathbb{A}^n$, until we have all the data of a ringed space. In later chapters, these ideas will grow up to become the core of modern algebraic geometry: the *scheme*.

§78.2 The Zariski topology on $\mathbb{A}^n$

Prototypical example for this section: In $\mathbb{A}^1$, closed sets are finite collections of points. In $\mathbb{A}^2$, a nonempty open set is the whole space minus some finite collection of curves/points.

We begin by endowing a topological structure on every variety V. Since our affine varieties (for now) all live in $\mathbb{A}^n$, all we have to do is put a suitable topology on $\mathbb{A}^n$, and then just view V as a subspace.

However, rather than putting the standard Euclidean topology on $\mathbb{A}^n$, we put a much more bizarre topology.

Definition 78.2.1. In the **Zariski topology** on $\mathbb{A}^n$, the *closed sets* are those of the form

$$\mathcal{V}(I) \qquad \text{where} \quad I \subseteq \mathbb{C}[x_1, \dots, x_n].$$

Of course, the open sets are complements of such sets.

Example 78.2.2 (Zariski topology on $\mathbb{A}^1$)

Let us determine the open sets of $\mathbb{A}^1$, which as usual we picture as a straight line (ignoring the fact that $\mathbb{C}$ is two-dimensional).

Since $\mathbb{C}[x]$ is a principal ideal domain, rather than looking at $\mathcal{V}(I)$ for every $I \subseteq \mathbb{C}[x]$, we just have to look at $\mathcal{V}(f)$ for a single f. There are a few flavors of polynomials f:

- The zero polynomial 0 which vanishes everywhere: this implies that the entire space $\mathbb{A}^1$ is a closed set.
- The constant polynomial 1 which vanishes nowhere. This implies that $\varnothing$ is a closed set.
- A polynomial $c(x-t_1)(x-t_2)\dots(x-t_n)$ of degree n. It has n roots, and so $\{t_1, \dots, t_n\}$ is a closed set.

Hence the closed sets of $\mathbb{A}^1$ are exactly all of $\mathbb{A}^1$ and finite sets of points (including $\varnothing$). Consequently, the *open* sets of $\mathbb{A}^1$ are

- $\varnothing$, and
- $\mathbb{A}^1$ minus a finite collection (possibly empty) of points.

Thus, the picture of a "typical" open set $\mathbb{A}^1$ might be

It's everything except a few marked points!

Example 78.2.3 (Zariski topology on $\mathbb{A}^2$)

Similarly, in $\mathbb{A}^2$, the interesting closed sets are going to consist of finite unions (possibly empty) of

- Closed curves, like $\mathcal{V}(y - x^2)$ (which is a parabola), and
- Single points, like $\mathcal{V}(x-3, y-4)$ (which is the point $(3,4)$).

Of course, the entire space $\mathbb{A}^2 = \mathcal{V}(0)$ and the empty set $\varnothing = \mathcal{V}(1)$ are closed sets.

Thus the nonempty open sets in $\mathbb{A}^2$ consist of the *entire* plane, minus a finite collection of points and one-dimensional curves.

Question 78.2.4. Draw a picture (to the best of your artistic ability) of a "typical" open set in $\mathbb{A}^2$.

All this is to say

The nonempty Zariski open sets are *huge*.

This is an important difference than what you're used to in topology. To be very clear:

- In the past, if I said something like "has so-and-so property in an open neighborhood of point p", one thought of this as saying "is true in a small region around p".
- In the Zariski topology, "has so-and-so property in an open neighborhood of point p" should be thought of as saying "is true for virtually all points, other than those on certain curves".

Indeed, "open neighborhood" is no longer really a accurate description. Nonetheless, in many pictures to follow, it will still be helpful to draw open neighborhoods as circles.

It remains to verify that as I've stated it, the closed sets actually form a topology. That is, I need to verify briefly that

- $\varnothing$ and $\mathbb{A}^n$ are both closed.
- Intersections of closed sets (even infinite) are still closed.
- Finite unions of closed sets are still closed.

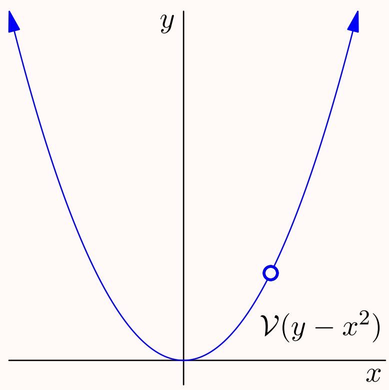
Well, closed sets are the same as affine varieties, so we already know this!

§78.3 The Zariski topology on affine varieties

Prototypical example for this section: If $V = \mathcal{V}(y - x^2)$ is a parabola, then V minus $(1,1)$ is open in V. Also, the plane minus the origin is $D(x) \cup D(y)$.

As we said before, by considering a variety V as a subspace of $\mathbb{A}^n$ it inherits the Zariski topology. One should think of an open subset of V as "V minus a few Zariski-closed sets". For example:

Example 78.3.1 (Open set of a variety)
Let $V = \mathcal{V}(y - x^2) \subseteq \mathbb{A}^2$ be a parabola, and let $U = V \setminus \{(1,1)\}$. We claim U is open in V.



Indeed, $\tilde{U} = \mathbb{A}^2 \setminus \{(1,1)\}$ is open in $\mathbb{A}^2$ (since it is the complement of the closed set $\mathcal{V}(x-1, y-1)$), so $U = \tilde{U} \cap V$ is open in V. Note that on the other hand the set U is *not* open in $\mathbb{A}^2$.

We will go ahead and introduce now a definition that will be very useful later.

Definition 78.3.2. Given $V \subseteq \mathbb{A}^n$ an affine variety and $f \in \mathbb{C}[x_1, \dots, x_n]$, we define the **distinguished open set** $D(f)$ to be the open set in V of points not vanishing on f:

$$D(f) = \{p \in V \mid f(p) \neq 0\} = V \setminus \mathcal{V}(f).$$

In [Va17], Vakil suggests remembering the notation $D(f)$ as "doesn't-vanish set".

Example 78.3.3 (Examples of (unions of) distinguished open sets)

(a) If $V = \mathbb{A}^1$ then $D(x)$ corresponds to a line minus a point.

(b) If $V = \mathcal{V}(y - x^2) \subseteq \mathbb{A}^2$, then $D(x-1)$ corresponds to the parabola minus $(1,1)$.

(c) If $V = \mathbb{A}^2$, then $D(x) \cup D(y) = \mathbb{A}^2 \setminus \{(0,0)\}$ is the punctured plane. You can show that this set is *not* distinguished open.

You can think of the concept as an analog to principal ideal: all open sets can be written in the form $V \setminus \mathcal{V}(I)$ for some ideal I, but if $I = (f)$ is principal then the set can be written as a distinguished open set $D(f)$. Similarly, the intersection of two distinguished open sets is distinguished, just as the product (not intersection!) of two principal ideals is principal.

Proposition 78.3.4 (Properties of distinguished open set)

Recall that $\mathcal{V}$ is inclusion-reversing, so being the complement of $\mathcal{V}$, we would expect D to be "inclusion-preserving". Indeed:

- If $(f) \subseteq (g)$ (that is, $g \mid f$), then $D(f) \subseteq D(g)$.
- Recall that $(fg) \subseteq (f) \cap (g)$. For distinguished open set, we have $D(fg) = D(f) \cap D(g)$.

It is useful to be familiar with the behavior of D.

Question 78.3.5. If $V = \mathcal{A}^2$, then $D(x)$ is the plane minus the y-axis, and $D(y)$ is the plane minus the x-axis. What is $D(xy)$?

§78.4 Coordinate rings

Prototypical example for this section: If $V = \mathcal{V}(y - x^2)$ then $\mathbb{C}[V] = \mathbb{C}[x,y]/(y - x^2)$.

The next thing we do is consider the functions from V to the base field $\mathbb{C}$. We restrict our attention to algebraic (polynomial) functions on a variety V: they should take every point $(a_1, \dots, a_n)$ on V to some complex number $P(a_1, \dots, a_n) \in \mathbb{C}$. For example, a valid function on a three-dimensional affine variety might be $(a, b, c) \mapsto a$; we just call this projection "x". Similarly we have a canonical projection y and z, and we can create polynomials by combining them, say $x^2y + 2xyz$.

Definition 78.4.1. The **coordinate ring** $\mathbb{C}[V]$ of a variety V is the ring of polynomial functions on V. (Notation explained next section.)

Remark 78.4.2 (Meaning of the name "coordinate ring") — We call the functions x, y and z above as the **coordinate functions**, as they maps each point in the variety V to its coordinate. So, the coordinate ring $\mathbb{C}[V]$ is simply the ring generated by $\mathbb{C}$ and the coordinate functions.

At first glance, we might think this is just $\mathbb{C}[x_1, \dots, x_n]$. But on closer inspection we realize that *on a given variety*, some of these functions are the same. For example, consider in $\mathbb{A}^2$ the parabola $V = \mathcal{V}(y - x^2)$. Then the two functions

$$\begin{aligned} V &\to \mathbb{C} \\ (x, y) &\mapsto x^2 \\ (x, y) &\mapsto y \end{aligned}$$

are actually the same function! We have to "mod out" by the ideal I which generates V. This leads us naturally to:

Theorem 78.4.3 (Coordinate rings correspond to ideal)
Let I be a radical ideal, and $V = \mathcal{V}(I) \subseteq \mathbb{A}^n$. Then

$$\mathbb{C}[V] \cong \mathbb{C}[x_1, \dots, x_n]/I.$$

Proof. There's a natural surjection as above

$$\mathbb{C}[x_1, \dots, x_n] \twoheadrightarrow \mathbb{C}[V]$$

and the kernel is I. $\square$

Thus properties of a variety V correspond to properties of the ring $\mathbb{C}[V]$.

§78.5 The sheaf of regular functions

Prototypical example for this section: Let $V = \mathbb{A}^1$, $U = V \setminus \{0\}$. Then $1/x \in \mathcal{O}_V(U)$ is regular on U.

Let V be an affine variety and let $\mathbb{C}[V]$ be its coordinate ring. As mentioned in the start of the chapter, we want to define a variety based on its intrinsic properties only, which is done by studying the collection of algebraic functions on it.

In [Va17] "Motivating example: The sheaf of differentiable functions" section, you can see a comparison of how a differentiable manifold can be studied by studying the differentiable functions on it.

Denote the set of all rational functions on V by $\mathcal{O}_V$ (as will be seen later, this terminology is not quite accurate as we need to allow multiple representations). We can view this as a set, however this does not capture the full structure of the rational functions:

Question 78.5.1. For any two elements f and g in $\mathbb{C}[V]$, show that the set where $\frac{f(x)}{g(x)}$ is well-defined is open in the Zariski topology. (Hint: $g^{\text{pre}}(0)$ is closed.)

So, we want to define a notion of $\mathcal{O}_V(U)$ for any open set U: the "nice" functions on any open subset. Obviously, any function in $\mathbb{C}[V]$ will work as a function on $\mathcal{O}_V(U)$.

However, to capture more of the structure we want to loosen our definition of "nice" function slightly by allowing *rational* functions.

The chief example is that $1/x$ should be a regular function on $\mathbb{A}^1 \setminus \{0\}$. The first natural guess is:

Definition 78.5.2. Let $U \subseteq V$ be an open set of the variety V. A **rational function** on U is a quotient $f(x)/g(x)$ of two elements f and g in $\mathbb{C}[V]$, where we require that $g(x) \neq 0$ for $x \in U$.

However, the definition is slightly too restrictive; we have to allow for multiple representations:

Definition 78.5.3. Let $U \subseteq V$ be open. We say a function $\phi \colon U \to \mathbb{C}$ is a **regular function** if for every point $p \in U$, we can find an open set $U_p \subseteq U$ containing p and a rational function f_p/g_p on U_p such that

$$\phi(x) = \frac{f_p(x)}{g_p(x)} \qquad \forall x \in U_p.$$

In particular, we require $g_p(x) \neq 0$ on the set U_p. We denote the set of all regular functions on U by $\mathcal{O}_V(U)$.

Thus,

ϕ is regular on U if it is locally a rational function.

This definition is misleadingly complicated, and the examples should illuminate it significantly. Firstly, in practice, most of the time we will be able to find a "global" representation of a regular function as a quotient, and we will not need to fuss with the p's. For example:

Example 78.5.4 (Regular functions)

(a) Any function in $f \in \mathbb{C}[V]$ is clearly regular, since we can take $g_p = 1$, $f_p = f$ for every p. So $\mathbb{C}[V] \subseteq \mathcal{O}_V(U)$ for any open set U.

(b) Let $V = \mathbb{A}^1$, $U_0 = V \setminus \{0\}$. Then $1/x \in \mathcal{O}_V(U_0)$ is regular on U_0.

(c) Let $V = \mathbb{A}^1$, $U_{12} = V \setminus \{1, 2\}$. Then

$$\frac{1}{(x-1)(x-2)} \in \mathcal{O}_V(U_{12})$$

is regular on U_{12}.

The "local" clause with p's is still necessary, though.

Example 78.5.5 (Requiring local representations)

Consider the variety

$$V = \mathcal{V}(ab - cd) \subseteq \mathbb{A}^4$$

and the open set $U = V \setminus \mathcal{V}(b, d)$. There is a regular function on U given by

$$(a, b, c, d) \mapsto \begin{cases} a/d & d \neq 0 \\ c/b & b \neq 0. \end{cases}$$

Clearly these are the "same function" (since $ab = cd$), but we cannot write "a/d" or "c/b" to express it because we run into divide-by-zero issues. That's why in the definition of a regular function, we have to allow multiple representations.

In fact, we will see later on that the definition of a regular function is a special case of a more general construction called *sheafification*, in which "presheaves of functions which are P" are transformed into "sheaves of functions which are *locally* P".

§78.6 Regular functions on distinguished open sets

Prototypical example for this section: Regular functions on $\mathbb{A}^1 \setminus \{0\}$ are $P(x)/x^n$.

The division-by-zero, as one would expect, essentially prohibits regular functions on the entire space V; i.e. there are no regular functions in $\mathcal{O}_V(V)$ that were not already in $\mathbb{C}[V]$. Actually, we have a more general result which computes the regular functions on distinguished open sets:

Theorem 78.6.1 (Regular functions on distinguished open sets)
Let $V \subseteq \mathbb{A}^n$ be an affine variety and $D(g)$ a distinguished open subset of it. Then

$$\mathcal{O}_V(D(g)) = \left\{ \frac{f}{g^k} \mid f \in \mathbb{C}[V] \text{ and } k \in \mathbb{Z} \right\}.$$

In particular, $\mathcal{O}_V(V) = \mathcal{O}_V(D(1)) \cong \mathbb{C}[V]$.

The proof of this theorem requires the Nullstellensatz, so it relies on $\mathbb{C}$ being algebraically closed. In fact, a counter-example is easy to find if we replace $\mathbb{C}$ by $\mathbb{R}$: consider $\frac{1}{x^2+1}$.

Proof. Obviously, every function of the form f/g^n works, so we want the reverse direction. This is long, and perhaps should be omitted on a first reading.

Here's the situation. Let $U = D(g)$. We're given a regular function ϕ, meaning at every point $p \in D(g)$, there is an open neighborhood U_p on which ϕ can be expressed as f_p/g_p (where $f_p, g_p \in \mathbb{C}[V]$). Then, we want to construct an $f \in \mathbb{C}[V]$ and an integer n such that $\phi = f/g^n$.

First, look at a particular U_p and f_p/g_p. Shrink U_p to a distinguished open set $D(h_p)$. Then, let $\widetilde{f_p} = f_p h_p$ and $\widetilde{g_p} = g_p h_p$. Thus we have that

$$\frac{\widetilde{f_p}}{\widetilde{g_p}} \text{ is correct on } D(h_p) \subseteq U \subseteq X.$$

The upshot of using the modified f_p and g_p is that:

$$\widetilde{f_p}\widetilde{g_q} = \widetilde{f_q}\widetilde{g_p} \qquad \forall p, q \in U.$$

Indeed, it is correct on $D(h_p) \cap D(h_q)$ by definition, and outside this set both the left-hand side and right-hand side are zero.

Now, we know that $D(g) = \bigcup_{p \in U} D(\widetilde{g_p})$, i.e.

$$\mathcal{V}(g) = \bigcap_{p \in U} \mathcal{V}(\widetilde{g_p}).$$

So by the Nullstellensatz we know that

$$g \in \sqrt{(\widetilde{g}_p : p \in U)} \implies \exists n : g^n \in (\widetilde{g}_p : p \in U).$$

In other words, for some n and $k_p \in \mathbb{C}[V]$ we have

$$g^n = \sum_p k_p \widetilde{g}_p$$

where only finitely many k_p are not zero. Now, we claim that

$$f := \sum_p k_p \widetilde{f}_p$$

works. This just observes by noting that for any $q \in U$, we have

$$f\widetilde{g}_q - g^n \widetilde{f}_q = \sum_p k_p(\widetilde{f}_p\widetilde{g}_q - \widetilde{g}_p\widetilde{f}_q) = 0. \qquad \square$$

This means that the *global* regular functions are just the same as those in the coordinate ring: you don't gain anything new by allowing it to be locally a quotient. (The same goes for distinguished open sets.)

Example 78.6.2 (Regular functions on distinguished open sets)

(a) As said already, taking $g = 1$ we recover $\mathcal{O}_V(V) \cong \mathbb{C}[V]$ for any affine variety V.

(b) Let $V = \mathbb{A}^1$, $U_0 = V \setminus \{0\}$. Then

$$\mathcal{O}_V(U_0) = \left\{ \frac{P(x)}{x^n} \mid P \in \mathbb{C}[x], \quad n \in \mathbb{Z} \right\}.$$

So more examples are $1/x$ and $(x+1)/x^3$.

Question 78.6.3. Why doesn't our theorem on regular functions apply to Example 78.5.5?

The regular functions will become of crucial importance once we define a scheme in the next chapter.

§78.7 Baby ringed spaces

In summary, given an affine variety V we have:

- A structure of a set of points,
- A structure of a topological space V on these points, and
- For every open set $U \subseteq V$, a ring $\mathcal{O}_V(U)$. Elements of the rings are functions $U \to \mathbb{C}$.

Let us agree that:

Definition 78.7.1. A **baby ringed space** is a topological space X equipped with a ring $\mathcal{O}_X(U)$ for every open set U. It is required that elements of the ring $\mathcal{O}_X(U)$ are functions $f \colon U \to \mathbb{C}$; we call these the *regular functions* of X on U.

Therefore, affine varieties are baby ringed spaces.

Remark 78.7.2 — This is not a standard definition. Hehe.

The reason this is called a "baby ringed space" is that in a *ringed space*, the rings $\mathcal{O}_V(U)$ can actually be *any rings*, but they have to satisfy a set of fairly technical conditions. When this happens, it's the $\mathcal{O}_V$ that does all the work; we think of $\mathcal{O}_V$ as a type of functor called a *sheaf*.

Since we are only studying affine/projective/quasi-projective varieties for the next chapters, we will just refer to these as baby ringed spaces so that we don't have to deal with the entire definition. The key concept is that we want to think of these varieties as *intrinsic objects*, free of any embedding. A baby ringed space is philosophically the correct thing to do.

Anyways, affine varieties are baby ringed spaces $(V, \mathcal{O}_V)$. In the next chapter we'll meet projective and quasi-projective varieties, which give more such examples of (baby) ringed spaces. With these examples in mind, we will finally lay down the complete definition of a ringed space, and use this to define a scheme.

§78.8 A few harder problems to think about

Problem 78A†. Show that for any $n \geq 1$ the Zariski topology of $\mathbb{A}^n$ is *not* Hausdorff.

Problem 78B†. Let V be an affine variety, and consider its Zariski topology.

(a) Show that the Zariski topology is **Noetherian**, meaning there is no infinite descending chain $Z_1 \supsetneq Z_2 \supsetneq Z_3 \supsetneq \dots$ of closed subsets.

(b) Prove that a Noetherian topological space is compact. Hence varieties are topologically compact.

Problem 78C* (Punctured Plane)**.** Let $V = \mathbb{A}^2$ and let $X = \mathbb{A}^2 \setminus \{(0,0)\}$ be the punctured plane (which is an open set of V). Compute $\mathcal{O}_V(X)$.

79 Projective varieties

Having studied affine varieties in $\mathbb{A}^n$, we now consider $\mathbb{CP}^n$. We will also make it into a baby ringed space in the same way as with $\mathbb{A}^n$.

§79.1 Graded rings

Prototypical example for this section: $\mathbb{C}[x_0, \dots, x_n]$ *is a graded ring.*

We first take the time to state what a graded ring is, just so that we have this language to use (now and later).

This definition is the same as Definition 76.3.2.

Definition 79.1.1. A **graded ring** R is a ring with the following additional structure: as an abelian group, it decomposes as

$$R = \bigoplus_{d \geq 0} R^d$$

where R^0, R^1, ..., are abelian groups. The ring multiplication has the property that if $r \in R^d$ and $s \in R^e$, we have $rs \in R^{d+e}$. Elements of an R^d are called **homogeneous elements**; we write "$d = \deg r$" to mean "$r \in R^d$".

We denote by R^+ the ideal $R \setminus R^0$ generated by the homogeneous elements of nonzero degree, and call it the **irrelevant ideal**.

Remark 79.1.2 — For experts: all our graded rings are commutative with 1.

Example 79.1.3 (Examples of graded rings)

(a) The ring $\mathbb{C}[x]$ is graded by degree: as abelian groups, $\mathbb{C}[x] \cong \mathbb{C} \oplus x\mathbb{C} \oplus x^2\mathbb{C} \oplus \dots$.

(b) More generally, the polynomial ring $\mathbb{C}[x_0, \dots, x_n]$ is graded by degree.

Abuse of Notation 79.1.4. The notation $\deg r$ is abusive in the case $r = 0$; note that $0 \in R^d$ for every d. So it makes sense to talk about "the" degree of r except when $r = 0$.

We will frequently refer to homogeneous ideals:

Definition 79.1.5. An ideal $I \subseteq \mathbb{C}[x_0, \dots, x_n]$ is **homogeneous** if it can be written as $I = (f_1, \dots, f_m)$ where each f_i is a homogeneous polynomial.

Remark 79.1.6 — If I and J are homogeneous, then so are $I + J$, IJ, $I \cap J$, $\sqrt{I}$.

Lemma 79.1.7 (Graded quotients are graded too)
Let I be a homogeneous ideal of a graded ring R. Then

$$R/I = \bigoplus_{d \geq 0} R^d/(R^d \cap I)$$

realizes R/I as a graded ring.

Since these assertions are just algebra, we omit their proofs here.

Remark 79.1.8 — In some other books, a homogeneous ideal (or **graded ideal**) is sometimes equivalently defined as an ideal I such that $I = \bigoplus_{d \geq 0}(R^d \cap I)$ as abelian group. In fact, we can verify that graded ideals are precisely the ones such that the quotient is naturally graded.

Example 79.1.9 (Example of a graded quotient ring)
Let $R = \mathbb{C}[x, y]$ and set $I = (x^3, y^2)$. Let $S = R/I$. Then

$$\begin{aligned} S^0 &= \mathbb{C} \\ S^1 &= \mathbb{C}x \oplus \mathbb{C}y \\ S^2 &= \mathbb{C}x^2 \oplus \mathbb{C}xy \\ S^3 &= \mathbb{C}x^2 y \\ S^d &= 0 \qquad \forall d \geq 4. \end{aligned}$$

So in fact $S = R/I$ is graded, and is a six-dimensional $\mathbb{C}$-vector space.

§79.2 The ambient space

Prototypical example for this section: Perhaps $\mathcal{V}_{pr}(x^2 + y^2 - z^2)$.

The set of points we choose to work with is $\mathbb{CP}^n$ this time, which for us can be thought of as the set of n-tuples

$$(x_0 : x_1 : \dots : x_n)$$

not all zero, up to scaling. Equivalently, it is the set of lines through the origin in $\mathbb{C}^{n+1}$. Projective space is defined in full in Section 64.6, and you should refer there if you aren't familiar with projective space.

The right way to think about it is "$\mathbb{A}^n$ plus points at infinity":

Definition 79.2.1. We define the set

$$U_i = \{(x_0 : \dots : x_n) \mid x_i \neq 0\} \subseteq \mathbb{CP}^n.$$

These are called the **standard affine charts**.

The name comes from:

Exercise 79.2.2 (Mandatory)**.** Give a natural bijection from U_i to $\mathbb{A}^n$. Thus we can think of $\mathbb{CP}^n$ as the affine set U_i plus "points at infinity".

Remark 79.2.3 — In fact, these charts U_i make $\mathbb{CP}^n$ with its usual topology into a complex manifold with holomorphic transition functions.

Example 79.2.4 (Colloquially, $\mathbb{CP}^1 = \mathbb{A}^1 \cup \{\infty\}$)
The space $\mathbb{CP}^1$ consists of pairs $(s : t)$, which you can think of as representing the complex number $z/1$. In particular $U_1 = \{(z : 1)\}$ is basically another copy of $\mathbb{A}^1$. There is only one new point, $(1 : 0)$.

However, like before we want to impose a Zariski topology on it. For concreteness, let's consider $\mathbb{CP}^2 = \{(x_0 : x_1 : x_2)\}$. We wish to consider zero loci in $\mathbb{CP}^2$, just like we did in affine space, and hence obtain a notion of a projective variety.

But this isn't so easy: for example, the function "x_0" is not a well-defined function on points in $\mathbb{CP}^2$ because $(x_0 : x_1 : x_2) = (5x_0 : 5x_1 : 5x_2)$! So we'd love to consider these "pseudo-functions" that still have zero loci. These are just the homogeneous polynomials f, because f is homogeneous of degree d if and only if

$$f(\lambda x_0, \dots, \lambda x_n) = \lambda^d f(x_0, \dots, x_n).$$

In particular, the relation "$f(x_0, \dots, x_n) = 0$" is well-defined if F is homogeneous. Thus, we can say:

Definition 79.2.5. If f is homogeneous, we can then define its **vanishing locus** as

$$\mathcal{V}_{\mathrm{pr}}(f) = \{(x_0 : \dots : x_n) \mid f(x_0, \dots, x_n) = 0\}.$$

The homogeneous condition is really necessary. For example, to require "$x_0 - 1 = 0$" makes no sense, since the points $(1 : 1 : 1)$ and $(2015 : 2015 : 2015)$ are the same.

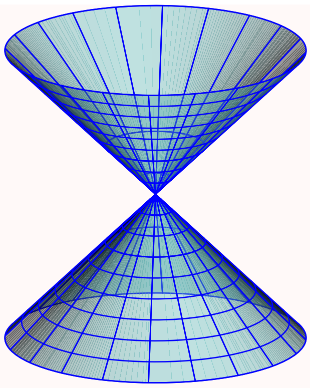
It's trivial to verify that homogeneous polynomials do exactly what we want; hence we can now define:

Definition 79.2.6. A **projective variety** in $\mathbb{CP}^n$ is the common zero locus of an arbitrary collection of homogeneous polynomials in $n+1$ variables.

Example 79.2.7 (A conic in $\mathbb{CP}^2$, or a cone in $\mathbb{C}^3$)
Let's try to picture the variety

$$\mathcal{V}_{\mathrm{pr}}(x^2 + y^2 - z^2) \subseteq \mathbb{CP}^2$$

which consists of the points $[x : y : z]$ such that $x^2 + y^2 = z^2$. If we view this as subspace of $\mathbb{C}^3$ (i.e. by thinking of $\mathbb{CP}^2$ as the set of lines through the origin), then we get a "cone":



If we take the standard affine charts now, we obtain:

- At $x = 1$, we get a hyperbola $\mathcal{V}(1 + y^2 - z^2)$.
- At $y = 1$, we get a hyperbola $\mathcal{V}(1 + x^2 - z^2)$.
- At $z = 1$, we get a circle $\mathcal{V}(x^2 + y^2 - 1)$.

That said, over $\mathbb{C}$ a hyperbola and circle are the same thing; I'm cheating a little by drawing $\mathbb{C}$ as one-dimensional, just like last chapter.

Question 79.2.8. Draw the intersection of the cone above with the $z = 1$ plane, and check that you do in fact get a circle. (This geometric picture will be crucial later.)

§79.3 Homogeneous ideals

Now, the next thing we want to do is define $\mathcal{V}_{\text{pr}}(I)$ for an ideal I. Of course, we again run into an issue with things like $x_0 - 1$ not making sense.

The way out of this is to use only *homogeneous* ideals.

Definition 79.3.1. If I is a homogeneous ideal, we define

$$\mathcal{V}_{\text{pr}}(I) = \{x \mid f(x) = 0 \; \forall f \in I\}.$$

Exercise 79.3.2. Show that the notion "$f(x) = 0 \; \forall f \in I$" is well-defined for a homogeneous ideal I.

So, we would hope for a Nullstellensatz-like theorem which bijects the homogeneous radical ideals to projective varieties. Unfortunately:

Example 79.3.3 (Irrelevant ideal)

To crush some dreams and hopes, consider the ideal

$$I = (x_0, x_1, \dots, x_n).$$

This is called the **irrelevant ideal**; it is a homogeneous radical yet $\mathcal{V}_{\text{pr}}(I) = \varnothing$.

However, other than the irrelevant ideal:

Theorem 79.3.4 (Homogeneous Nullstellensatz)
Let I and J be homogeneous ideals.

(a) If $\mathcal{V}_{\text{pr}}(I) = \mathcal{V}_{\text{pr}}(J) \neq \varnothing$ then $\sqrt{I} = \sqrt{J}$.

(b) If $\mathcal{V}_{\text{pr}}(I) = \varnothing$, then either $I = (1)$ or $\sqrt{I} = (x_0, x_1, \dots, x_n)$.

Thus there is a natural bijection between:

- projective varieties in $\mathbb{CP}^n$, and
- homogeneous radical ideals of $\mathbb{C}[x_0, \dots, x_n]$ except for the irrelevant ideal.

Proof. For the first part, let $V = \mathcal{V}_{\text{pr}}(I)$ and $W = \mathcal{V}_{\text{pr}}(J)$ be projective varieties in $\mathbb{CP}^n$. We can consider them as *affine varieties* in $\mathbb{A}^{n+1}$ by using the interpretation of $\mathbb{CP}^n$ as lines through the origin in $\mathbb{C}^n$.

Algebraically, this is done by taking the homogeneous ideals $I, J \subseteq \mathbb{C}[x_0, \dots, x_n]$ and using the same ideals to cut out *affine* varieties $V_{\text{aff}} = \mathcal{V}(I)$ and $W_{\text{aff}} = \mathcal{V}(J)$ in $\mathbb{A}^{n+1}$. For example, the cone $x^2 + y^2 - z^2 = 0$ is a conic (a one-dimensional curve) in $\mathbb{CP}^2$, but can also be thought of as a cone (which is a two-dimensional surface) in $\mathbb{A}^3$.

Then for (a), we have $V_{\text{aff}} = W_{\text{aff}}$, so $\sqrt{I} = \sqrt{J}$.

For (b), either V_{aff} is empty or it is just the origin of $\mathbb{A}^{n+1}$, so the Nullstellensatz implies either $I = (1)$ or $\sqrt{I} = (x_0, \dots, x_n)$ as desired. $\square$

Projective analogues of Theorem 77.4.2 (on intersections and unions of varieties) hold verbatim for projective varieties as well.

§79.4 As ringed spaces

Prototypical example for this section: The regular functions on $\mathbb{CP}^1$ minus a point are exactly those of the form $P(s/t)$.

Now, let us make every projective variety V into a baby ringed space. We already have the set of points, a subset of $\mathbb{CP}^n$.

The topology is defined as follows.

Definition 79.4.1. We endow $\mathbb{CP}^n$ with the **Zariski topology** by declaring the sets of the form $\mathcal{V}_{\text{pr}}(I)$, where I is a homogeneous ideal, to be the closed sets.

Every projective variety V then inherits the Zariski topology from its parent $\mathbb{CP}^n$. The **distinguished open sets** $D(f)$ are $V \setminus \mathcal{V}_{\text{pr}}(f)$.

Thus every projective variety V is now a topological space. It remains to endow it with a sheaf of regular functions $\mathcal{O}_V$. To do this we have to be a little careful. In the affine case we had a nice little ring of functions, the coordinate ring $\mathbb{C}[x_0, \dots, x_n]/I$, that we could use to provide the numerator and denominators. So, it seems natural to then define:

Definition 79.4.2. The **homogeneous coordinate ring** of a projective variety $V = \mathcal{V}_{\text{pr}}(I) \subseteq \mathbb{CP}^n$, where I is homogeneous radical, is defined as the ring

$$\mathbb{C}[V] = \mathbb{C}[x_0, \dots, x_n]/I.$$

Remark 79.4.3 — Unlike the case of Remark 78.4.2, an element of $\mathbb{C}[V]$ no longer correspond to a function from V to $\mathbb{C}$; nevertheless, it is a function from $\mathcal{V}(I) \subseteq \mathbb{A}^{n+1}$ to $\mathbb{C}$.

However, when we define a rational function we must impose a new requirement that the numerator and denominator are the same degree.

Definition 79.4.4. Let $U \subseteq V$ be an open set of a projective variety V. A **rational function** ϕ on a projective variety V is a quotient f/g, where $f, g \in \mathbb{C}[V]$, and f and g are homogeneous of the same degree, and $\mathcal{V}_{\text{pr}}(g) \cap U = \varnothing$. In this way we obtain a function $\phi\colon U \to \mathbb{C}$.

Example 79.4.5 (Examples of rational functions)
Let $V = \mathbb{CP}^1$ have coordinates $(s : t)$.

(a) If $U = V$, then constant functions $c/1$ are the only rational functions on U.

(b) Now let $U_1 = V \setminus \{(1:0)\}$. Then, an example of a regular function is

$$\frac{s^2 + 9t^2}{t^2} = \left(\frac{s}{t}\right)^2 + 9.$$

If we think of U_1 as $\mathbb{C}$ (i.e. $\mathbb{CP}^1$ minus an infinity point, hence like $\mathbb{A}^1$) then really this is just the function $x^2 + 9$.

Then we can repeat the same definition as before:

Definition 79.4.6. Let $U \subseteq V$ be an open set of a projective variety V. We say a function $\phi\colon U \to \mathbb{C}$ is a **regular function** if for every point p, we can find an open set U_p containing p and a rational function f_p/g_p on U_p such that

$$\phi(x) = \frac{f_p(x)}{g_p(x)} \qquad \forall x \in U_p.$$

In particular, we require $U_p \cap \mathcal{V}_{\text{pr}}(g_p) = \varnothing$. We denote the set of all regular functions on U by $\mathcal{O}_V(U)$.

Of course, the rational functions from the previous example are examples of regular functions as well. This completes the definition of a projective variety V as a baby ringed space.

§79.5 Examples of regular functions

Naturally, I ought to tell you what the regular functions on distinguished open sets are; this is an analog to Theorem 78.6.1 from last time.

Theorem 79.5.1 (Regular functions on distinguished open sets for projective varieties)
Let V be a projective variety, and let $g \in \mathbb{C}[V]$ be homogeneous of *positive degree* (thus g is nonconstant). Then

$$\mathcal{O}_V(D(g)) = \left\{ \frac{f}{g^r} \mid f \in \mathbb{C}[V] \text{ homogeneous of degree } r \deg g \right\}.$$

What about the case $g = 1$? A similar result holds, but we need the assumption that V is irreducible.

Definition 79.5.2. A projective variety V is irreducible if it can't be written as the union of two proper (projective) sub-varieties.

Theorem 79.5.3 (Only constant regular functions on projective space)
Let V be an *irreducible* projective variety. Then the only regular functions on V are constant, thus we have
$$\mathcal{O}_V(V) \cong \mathbb{C}.$$
This relies on the fact that $\mathbb{C}$ is algebraically closed.

Proofs of these are omitted for now.

Example 79.5.4 (Irreducibility is needed above)
The reason we need V irreducible is otherwise we could, for example, take V to be the union of two points; in this case $\mathcal{O}_V(V) \cong \mathbb{C}^{\oplus 2}$.

Remark 79.5.5 — It might seem strange that $\mathcal{O}_V(D(g))$ behaves so differently when $g = 1$. One vague explanation is that in a projective variety, a distinguished open $D(g)$ looks much like an affine variety if $\deg g > 0$. For example, in $\mathbb{CP}^1$ we have $\mathbb{CP}^1 \setminus \{0\} \cong \mathbb{A}^1$ (where $\cong$ is used in a sense that I haven't made precise). Thus the claim becomes related to the corresponding affine result. But if $\deg g = 0$ and $g \neq 0$, then $D(g)$ is the entire projective variety, which does not look affine, and thus the analogy breaks down.

Example 79.5.6 (Regular functions on $\mathbb{CP}^1$)
Let $V = \mathbb{CP}^1$, with coordinates $(s : t)$.

(a) By Theorem 79.5.1, if U_1 is the standard affine chart omitting the point $(1 : 0)$, we have $\mathcal{O}_V(U_1) = \left\{ \frac{f}{t^n} \mid \deg f = n \right\}$. One can write this as
$$\mathcal{O}_V(U_1) \cong \{P(s/t) \mid P \in \mathbb{C}[x]\} \cong \mathcal{O}_{\mathbb{A}^1}(\mathbb{A}^1).$$
This conforms with our knowledge that U_1 "looks very much like $\mathbb{A}^1$".

(b) As V is irreducible, $\mathcal{O}_V(V) = \mathbb{C}$: there are no nonconstant functions on $\mathbb{CP}^1$.

Example 79.5.7 (Regular functions on $\mathbb{CP}^2$)
Let $\mathbb{CP}^2$ have coordinates $(x : y : z)$ and let $U_0 = \left\{ (x : y : 1) \in \mathbb{CP}^2 \right\}$ be the

distinguished open set $D(z)$. Then in the same vein,

$$\mathcal{O}_{\mathbb{CP}^2}(U_0) = \left\{ \frac{P(x,y)}{z^n} \mid \deg P = n \right\} \cong \left\{ P(x/z, y/z) \mid P \in \mathbb{C}[x,y] \right\}.$$

§79.6 A few harder problems to think about

Problems:

80 Bonus: Bézout's theorem

In this chapter we discuss Bézout's theorem. It makes precise the idea that two degree d and e curves in $\mathbb{CP}^2$ should intersect at "exactly" de points. (We work in projective space so e.g. any two lines intersect.)

§80.1 Non-radical ideals

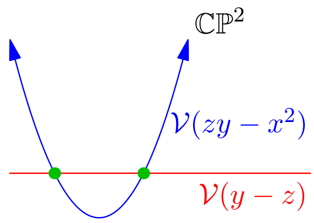
Prototypical example for this section: Tangent to the parabola.

We need to account for multiplicities. So we will whenever possible work with homogeneous ideals I, rather than varieties V, because we want to allow the possibility that I is not radical. Let's see how we might do so.

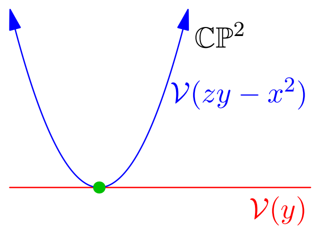
For a first example, suppose we intersect $y = x^2$ with the line $y = 1$; or more accurately, in projective coordinates of $\mathbb{CP}^2$, the parabola $zy = x^2$ and $y = z$. The ideal of the intersection is

$$(zy - x^2, y - z) = (x^2 - z^2, y - z) \subseteq \mathbb{C}[x, y, z].$$

So this corresponds to having two points; this gives two intersection points: $(1 : 1 : 1)$ and $(-1 : 1 : 1)$. Here is a picture of the two varieties in the affine $z = 1$ chart:



That's fine, but now suppose we intersect $zy = x^2$ with the line $y = 0$ instead. Then we instead get a "double point":



The corresponding ideal is this time

$$(zy - x^2, y) = (x^2, y) \subseteq \mathbb{C}[x, y, z].$$

This ideal is *not* radical, and when we take $\sqrt{(x^2, y)} = (x, y)$ we get the ideal which corresponds to a single projective point $(0 : 0 : 1)$ of $\mathbb{CP}^2$. This is why we work with ideals rather than varieties: we need to tell the difference between (x^2, y) and (x, y).

§80.2 Hilbert functions of finitely many points

Prototypical example for this section: The Hilbert function attached to the double point (x^2, y) is eventually the constant 2.

Definition 80.2.1. Given a nonempty projective variety V, there is a unique radical ideal I such that $V = \mathcal{V}_{\text{pr}}(I)$. In this chapter we denote it by $\mathcal{I}_{\text{rad}}(V)$. For an empty variety we set $\mathcal{I}_{\text{rad}}(\varnothing) = (1)$, rather than choosing the irrelevant ideal.

Definition 80.2.2. Let $I \subseteq \mathbb{C}[x_0, \dots, x_n]$ be homogeneous. We define the **Hilbert function** of I, denoted $h_I : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ by

$$h_I(d) = \dim_{\mathbb{C}} \left(\mathbb{C}[x_0, \dots, x_n]/I\right)^d$$

i.e. $h_I(d)$ is the dimension of the dth graded part of $\mathbb{C}[x_0, \dots, x_n]/I$.

Definition 80.2.3. If V is a projective variety, we set $h_V = h_{\mathcal{I}_{\text{rad}}(V)}$, where $I = \mathcal{I}_{\text{rad}}(V)$ is the *radical* ideal satisfying $V = \mathcal{V}_{\text{pr}}(I)$ as defined above.

In this case, $\mathbb{C}[x_0, \dots, x_n]/I$ is just $\mathbb{C}[V]$.

Example 80.2.4 (Examples of Hilbert functions in zero dimensions)

For concreteness, let us use $\mathbb{CP}^2$.

(a) If V is the single point $(0 : 0 : 1)$, with ideal $\mathcal{I}_{\text{rad}}(V) = (x, y)$, then

$$\mathbb{C}[V] = \mathbb{C}[x, y, z]/(x, y) \cong \mathbb{C}[z] \cong \mathbb{C} \oplus z\mathbb{C} \oplus z^2\mathbb{C} \oplus z^3\mathbb{C} \dots$$

which has dimension 1 in all degrees. Consequently, we have

$$h_I(d) \equiv 1.$$

(b) Now suppose we use the "double point" ideal $I = (x^2, y)$. This time, we have

$$\begin{aligned} \mathbb{C}[x, y, z]/(x^2, y) &\cong \mathbb{C}[z] \oplus x\mathbb{C}[z] \\ &\cong \mathbb{C} \oplus (x\mathbb{C} \oplus z\mathbb{C}) \oplus (xz\mathbb{C} \oplus z^2\mathbb{C}) \oplus (xz^2\mathbb{C} \oplus z^3\mathbb{C}) \oplus \dots . \end{aligned}$$

From this we deduce that

$$h_I(d) = \begin{cases} 2 & d = 1, 2, 3, \dots \\ 1 & d = 0. \end{cases}$$

(c) Let's now take the variety $V = \{(1 : 1 : 1), (-1 : 1 : 1)\}$ consisting of two points, with $\mathcal{I}_{\text{rad}}(V) = (x^2 - z^2, y - z)$. Then

$$\begin{aligned} \mathbb{C}[x, y, z]/(x^2 - z^2, y - z) &\cong \mathbb{C}[x, z]/(x^2 - z^2) \\ &\cong \mathbb{C}[z] \oplus x\mathbb{C}[z]. \end{aligned}$$

So this example has the same Hilbert function as the previous one.

Abuse of Notation 80.2.5. I'm abusing the isomorphism symbol $\mathbb{C}[z] \cong \mathbb{C} \oplus z\mathbb{C} \oplus z^2\mathbb{C}$ and similarly in other examples. This is an isomorphism only on the level of $\mathbb{C}$-vector spaces. However, in computing Hilbert functions of other examples I will continue using this abuse of notation.

Example 80.2.6 (Hilbert functions for empty varieties)

Suppose $I \subsetneq \mathbb{C}[x_0, \dots, x_n]$ is an ideal, possibly not radical but such that

$$\mathcal{V}_{\text{pr}}(I) = \varnothing$$

hence $\sqrt{I} = (x_0, \dots, x_n)$ is the irrelevant ideal. Thus there are integers d_i for $i = 0, \dots, n$ such that $x_i^{d_i} \in I$ for every i; consequently, $h_I(d) = 0$ for any $d > d_0 + \dots + d_n$. We summarize this by saying that

$$h_I(d) = 0 \text{ for all } d \gg 0.$$

Here the notation $d \gg 0$ means "all sufficiently large d".

From these examples we see that if I is an ideal, then the Hilbert function appears to eventually be constant, with the desired constant equal to the size of $\mathcal{V}_{\text{pr}}(I)$, "with multiplicity" in the case that I is not radical.

Let's prove this. Before proceeding we briefly remind the reader of short exact sequences: a sequence of maps of $0 \to V \hookrightarrow W \twoheadrightarrow X \to 0$ is one such that the $\operatorname{im}(V \hookrightarrow W) = \ker(W \twoheadrightarrow X)$ (and of course the maps $V \hookrightarrow W$ and $W \twoheadrightarrow X$ are injective and surjective). If V, W, X are finite-dimensional vector spaces over $\mathbb{C}$ this implies that $\dim W = \dim V + \dim X$.

Proposition 80.2.7 (Hilbert functions of $I \cap J$ and $I + J$)

Let I and J be homogeneous ideals in $\mathbb{C}[x_0, \dots, x_n]$. Then

$$h_{I \cap J} + h_{I+J} = h_I + h_J.$$

Proof. Consider any $d \geq 0$. Let $S = \mathbb{C}[x_0, \dots, x_n]$ for brevity. Then

$$\begin{tikzcd}
0 \arrow[r] & {[S/(I \cap J)]^d} \arrow[r, hook] & {[S/I]^d \oplus [S/J]^d} \arrow[r, two heads] & {[S/(I+J)]^d} \arrow[r] & 0 \\
 & f \arrow[r, mapsto] & (f,f) & & \\
 & & (f,g) \arrow[r, mapsto] & f-g &
\end{tikzcd}$$

is a short exact sequence of vector spaces. Therefore, for every $d \geq 0$ we have that

$$\dim\,[S/I]^d \oplus [S/J]^d = \dim\,[S/(I \cap J)]^d + \dim\,[S/(I+J)]^d$$

which gives the conclusion. $\square$

Example 80.2.8 (Hilbert function of two points in $\mathbb{CP}^1$)

In $\mathbb{CP}^1$ with coordinate ring $\mathbb{C}[s,t]$, consider $I = (s)$ the ideal corresponding to the point $(0 : 1)$ and $J = (t)$ the ideal corresponding to the point $(1 : 0)$. Then $I \cap J = (st)$ is the ideal corresponding to the disjoint union of these two points, while $I + J = (s,t)$ is the irrelevant ideal. Consequently $h_{I+J}(d) = 0$ for $d \gg 0$. Therefore, we get

$$h_{I \cap J}(d) = h_I(d) + h_J(d) \text{ for } d \gg 0$$

so the Hilbert function of a two-point projective variety is the constant 2 for $d \gg 0$.

This example illustrates the content of the main result:

Theorem 80.2.9 (Hilbert functions of zero-dimensional varieties)
Let V be a projective variety consisting of m points (where $m \ge 0$ is an integer). Then

$$h_V(d) = m \text{ for } d \gg 0.$$

Proof. We already did $m = 0$, so assume $m \ge 1$. Let $I = \mathcal{I}_{\text{rad}}(V)$ and for $k = 1, \dots, m$ let $I_k = \mathcal{I}_{\text{rad}}(k\text{th point of } V)$.

Exercise 80.2.10. Show that $h_{I_k}(d) = 1$ for every d. (Modify Example 80.2.4(a).)

Hence we can proceed by induction on $m \ge 2$, with the base case $m = 1$ already done above. For the inductive step, we use the projective analogues of Theorem 77.4.2. We know that $h_{I_1 \cap \dots \cap I_{m-1}}(d) = m - 1$ for $d \gg 0$ (this is the first $m-1$ points; note that $I_1 \cap \dots \cap I_{m-1}$ is radical). To add in the mth point we note that

$$h_{I_1 \cap \dots \cap I_m}(d) = h_{I_1 \cap \dots I_{m-1}}(d) + h_{I_m}(d) - h_J(d)$$

where $J = (I_1 \cap \dots \cap I_{m-1}) + I_m$. The ideal J may not be radical, but satisfies $\mathcal{V}_{\text{pr}}(J) = \varnothing$ by an earlier example, hence $h_J = 0$ for $d \gg 0$. This completes the proof. $\square$

In exactly the same way we can prove that:

Corollary 80.2.11 (h_I eventually constant when $\dim \mathcal{V}_{\text{pr}}(I) = 0$)
Let I be an ideal, not necessarily radical, such that $\mathcal{V}_{\text{pr}}(I)$ consists of finitely many points. Then the Hilbert h_I is eventually constant.

Proof. Induction on the number of points, $m \ge 1$. The base case $m = 1$ was essentially done in Example 80.2.4(b) and Exercise 80.2.10. The inductive step is literally the same as in the proof above, except no fuss about radical ideals. $\square$

§80.3 Hilbert polynomials

So far we have only talked about Hilbert functions of zero-dimensional varieties, and showed that they are eventually constant. Let's look at some more examples.

Example 80.3.1 (Hilbert function of $\mathbb{CP}^n$)
The Hilbert function of $\mathbb{CP}^n$ is

$$h_{\mathbb{CP}^n}(d) = \binom{d+n}{n} = \frac{1}{n!}(d+n)(d+n-1)\dots(d+1)$$

by a "balls and urns" argument. This is a polynomial of degree n.

Example 80.3.2 (Hilbert function of the parabola)
Consider the parabola $zy - x^2$ in $\mathbb{CP}^2$ with coordinates $\mathbb{C}[x, y, z]$. Then

$$\mathbb{C}[x, y, z]/(zy - x^2) \cong \mathbb{C}[y, z] \oplus x\mathbb{C}[y, z].$$

A combinatorial computation gives that

$$\begin{aligned} h_{(zy-x^2)}(0) &= 1 & \text{Basis } 1 \\ h_{(zy-x^2)}(1) &= 3 & \text{Basis } x, y, z \\ h_{(zy-x^2)}(2) &= 5 & \text{Basis } xy,\ xz,\ y^2,\ yz,\ z^2. \end{aligned}$$

We thus in fact see that $h_{(zy-x^2)}(d) = 2d - 1$.

In fact, this behavior of "eventually polynomial" always works.

Theorem 80.3.3 (Hilbert polynomial)
Let $I \subseteq \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous ideal, not necessarily radical. Then

(a) There exists a polynomial χ_I such that $h_I(d) = \chi_I(d)$ for all $d \gg 0$.

(b) $\deg \chi_I = \dim \mathcal{V}_{\mathrm{pr}}(I)$ (if $\mathcal{V}_{\mathrm{pr}}(I) = \varnothing$ then $\chi_I = 0$).

(c) The polynomial $m! \cdot \chi_I$ has integer coefficients.

Proof. The base case was addressed in the previous section.

For the inductive step, consider $\mathcal{V}_{\mathrm{pr}}(I)$ with dimension m. Consider a hyperplane H such that no irreducible component of $\mathcal{V}_{\mathrm{pr}}(I)$ is contained inside H (we quote this fact without proof, as it is geometrically obvious, but the last time I tried to write the proof I messed up). For simplicity, assume WLOG that $H = \mathcal{V}_{\mathrm{pr}}(x_0)$.

Let $S = \mathbb{C}[x_0, \dots, x_n]$ again. Now, consider the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & {[S/I]^{d-1}} \arrow[r, hook] & {[S/I]^d} \arrow[r, two heads] & {[S/(I + (x_0))]^d} \arrow[r] & 0 \\ & f \arrow[r, maps to] & f \cdot x_0 & & \\ & & f \arrow[r, maps to] & f. & \end{tikzcd}$$

(The injectivity of the first map follows from the assumption about irreducible components of $\mathcal{V}_{\mathrm{pr}}(I)$.) Now exactness implies that

$$h_I(d) - h_I(d-1) = h_{I+(x_0)}(d).$$

The last term geometrically corresponds to $\mathcal{V}_{\mathrm{pr}}(I) \cap H$; it has dimension $m - 1$, so by the inductive hypothesis we know that

$$h_I(d) - h_I(d-1) = \frac{c_0 d^{m-1} + c_1 d^{m-2} + \cdots + c_{m-1}}{(m-1)!} \qquad d \gg 0$$

for some integers $c_0, \dots, c_{m-1}$. Then we are done by the theory of **finite differences** of polynomials. $\square$

§80.4 Bézout's theorem

Definition 80.4.1. We call χ_I the **Hilbert polynomial** of I. If χ_I is nonzero, we call the leading coefficient of $m!\chi_I$ the **degree** of I, which is an integer, denoted $\deg I$.

Of course for projective varieties V we let $h_V = h_{\mathcal{I}_{\text{rad}}(V)}$, and $\deg V = \deg \mathcal{I}_{\text{rad}}(V)$.

Remark 80.4.2 — Note that the degree of an ideal $\deg I$ is not the same as $\deg h_I$!

Let us show some properties of the degrees, which will allow us to compute the degree of any projective variety from its irreducible components.

Proposition 80.4.3 (Properties of degrees)

For two varieties V and W, we have the following:

- If V and W are disjoint and have the same dimension, then $\deg(V \cup W) = \deg V + \deg W$.
- If $\dim V < \dim W$, then $\deg(V \cup W) = \deg W$.

So,

The degree is additive over components, and it measures the "degree" of the highest-dimensional component.

Proof. Follows from the properties of Hilbert polynomial in Theorem 80.3.3 and Proposition 80.2.7, and that the leading coefficient only depends on the largest-degree summand. □

Example 80.4.4 (Examples of degrees)

(a) If V is a finite set of $n \geq 1$ points, it has degree n.

(b) If I corresponds to a double point, it has degree 2.

(c) $\mathbb{CP}^n$ has degree 1.

(d) Any line or plane, being "isomorphic" to $\mathbb{CP}^1$ and $\mathbb{CP}^2$ respectively, has degree 1.

(e) The parabola has degree 2. (Note that, as an algebraic variety, the parabola is isomorphic to a line!)

(f) The union of the parabola and a point has degree 2.

Now, you might guess that if f is a homogeneous quadratic polynomial then the degree of the principal ideal (f) is 2, and so on. (Thus for example we expect a circle to have degree 2.) This is true:

Theorem 80.4.5 (Bézout's theorem)

Let I be a homogeneous ideal of $\mathbb{C}[x_0, \dots, x_n]$, such that $\dim \mathcal{V}_{\text{pr}}(I) \geq 1$. Let $f \in \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous polynomial of degree k which does not vanish on any irreducible component of $\mathcal{V}_{\text{pr}}(I)$. Then

$$\deg\left(I + (f)\right) = k \deg I.$$

Geometrically,

If V is any projective variety, $\mathcal{V}(f)$ is a hyperplane of degree k, then their intersection $V \cap \mathcal{V}(f)$ has degree $k \deg V$ — unless some irreducible component of V is contained inside $\mathcal{V}(f)$.

This is what we mentioned at the beginning of the chapter.

Because the ideal I may not be radical, the geometric interpretation statement is not the most general possible — the problem will be rectified later with the generalization to schemes.

Proof. Let $S = \mathbb{C}[x_0, \dots, x_n]$ again. This time the exact sequence is

$$0 \longrightarrow [S/I]^{d-k} \hookrightarrow [S/I]^d \twoheadrightarrow [S/(I + (f))]^d \longrightarrow 0.$$

We leave this olympiad-esque exercise as Problem 80A. □

§80.5 Applications

First, we show that the notion of degree is what we expect.

Corollary 80.5.1 (Hypersurfaces: the degree deserves its name)

Let V be a hypersurface, i.e. $\mathcal{I}_{\text{rad}}(V) = (f)$ for f a homogeneous polynomial of degree k. Then $\deg V = k$.

Proof. Recall $\deg(0) = \deg \mathbb{CP}^n = 1$. Take $I = (0)$ in Bézout's theorem. □

The common special case in $\mathbb{CP}^2$ is:

Corollary 80.5.2 (Bézout's theorem for curves)

For any two curves X and Y in $\mathbb{CP}^2$ without a common irreducible component,

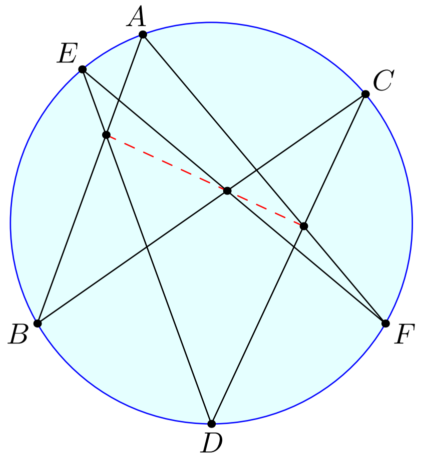
$$|X \cap Y| \leq \deg X \cdot \deg Y.$$

Now, we use this to prove Pascal's theorem.

Theorem 80.5.3 (Pascal's theorem)

Let A, B, C, D, E, F be six distinct points which lie on a conic $\mathscr{C}$ in $\mathbb{CP}^2$. Then the points $AB \cap DE$, $BC \cap EF$, $CD \cap FA$ are collinear.

Proof. Let X be the variety equal to the union of the three lines AB, CD, EF, hence $X = \mathcal{V}_{\mathrm{pr}}(f)$ for some cubic polynomial f (which is the product of three linear ones). Similarly, let $Y = \mathcal{V}_{\mathrm{pr}}(g)$ be the variety equal to the union of the three lines BC, DE, FA.



Now let P be an arbitrary point on the conic on $\mathscr{C}$, distinct from the six points A, B, C, D, E, F. Consider the projective variety

$$V = \mathcal{V}_{\mathrm{pr}}(\alpha f + \beta g)$$

where the constants α and β are chosen such that $P \in V$.

Question 80.5.4. Show that V also contains the six points A, B, C, D, E, F as well as the three points $AB \cap DE$, $BC \cap EF$, $CD \cap FA$ regardless of which α and β are chosen.

Now, note that $|V \cap \mathscr{C}| \geq 7$. But $\deg V = 3$ and $\deg \mathscr{C} = 2$. This contradicts Bézout's theorem unless V and $\mathscr{C}$ share an irreducible component. This can only happen if V is the union of a line and conic, for degree reasons; i.e. we must have that

$$V = \mathscr{C} \cup \text{line}.$$

Finally note that the three intersection points $AB \cap DE$, $BC \cap EF$ and $CD \cap FA$ do not lie on $\mathscr{C}$, so they must lie on this line. $\square$

We'd like to remark that the Pascal's theorem is just a special case of the Cayley-Bacharach theorem, which can be used to prove that the addition operation on an elliptic curve is associative. Interested readers may want to try proving the Cayley-Bacharach theorem using the same technique.

§80.6 A few harder problems to think about

Problem 80A. Complete the proof of Bézout's theorem from before.

Problem 80B (USA TST 2016/6)**.** Let ABC be an acute scalene triangle and let P be a point in its interior. Let A_1, B_1, C_1 be projections of P onto triangle sides BC, CA, AB, respectively. Find the locus of points P such that AA_1, BB_1, CC_1 are concurrent and $\angle PAB + \angle PBC + \angle PCA = 90°$.

81 Morphisms of varieties

In preparation for our work with schemes, we will finish this part by talking about *morphisms* between affine and projective varieties, given that we have taken the time to define them.

Idea: we know both affine and projective varieties are special cases of baby ringed spaces, so in fact we will just define a morphism between *any* two baby ringed spaces.

§81.1 Defining morphisms of baby ringed spaces

Prototypical example for this section: See next section.

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be baby ringed spaces, and think about how to define a morphism between them.

The guiding principle in algebra is that we want morphisms to be functions on underlying structure, but also respect the enriched additional data on top. To give some examples from the very beginning of time:

Example 81.1.1 (How to define a morphism)

- Consider groups. A group G has an underlying set (of elements), which we then enrich with a multiplication operation. So a homomorphism is a map of the underlying sets, plus it has to respect the group multiplication.
- Consider R-modules. Each R-module has an underlying abelian group, which we then enrich with scalar multiplication. So we require that a linear map respects the scalar multiplication as well, in addition to being a homomorphism of abelian groups.
- Consider topological spaces. A space X has an underlying set (of points), which we then enrich with a topology of open sets. So we consider maps of the set of points which respect the topology (pre-images of open sets are open).

This time, the ringed spaces $(X, \mathcal{O}_X)$ have an underlying *topological space*, which we have enriched with a structure sheaf. So, we want a continuous map $f \colon X \to Y$ of these topological spaces, which we then need to respect the sheaf of regular functions.

How might we do this? Well, if we let $\psi \colon Y \to \mathbb{C}$ be a regular function, then composition gives a natural way to write a map $X \to Y \to \mathbb{C}$. We then want to require that this is also a regular function.

More generally, we can take any regular function on Y and obtain some function on X, which we call a pullback. We then require that all the pullbacks are regular on X.

Definition 81.1.2. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be baby ringed spaces. Given a map $f \colon X \to Y$ and a regular function $\phi \in \mathcal{O}_Y(U)$, we define the **pullback** of ϕ, denoted $f^\sharp \phi$, to be the composed function

$$f^{\text{pre}}(U) \xrightarrow{f} U \xrightarrow{\phi} \mathbb{C}.$$

The use of the word "pullback" is the same as in our study of differential forms.

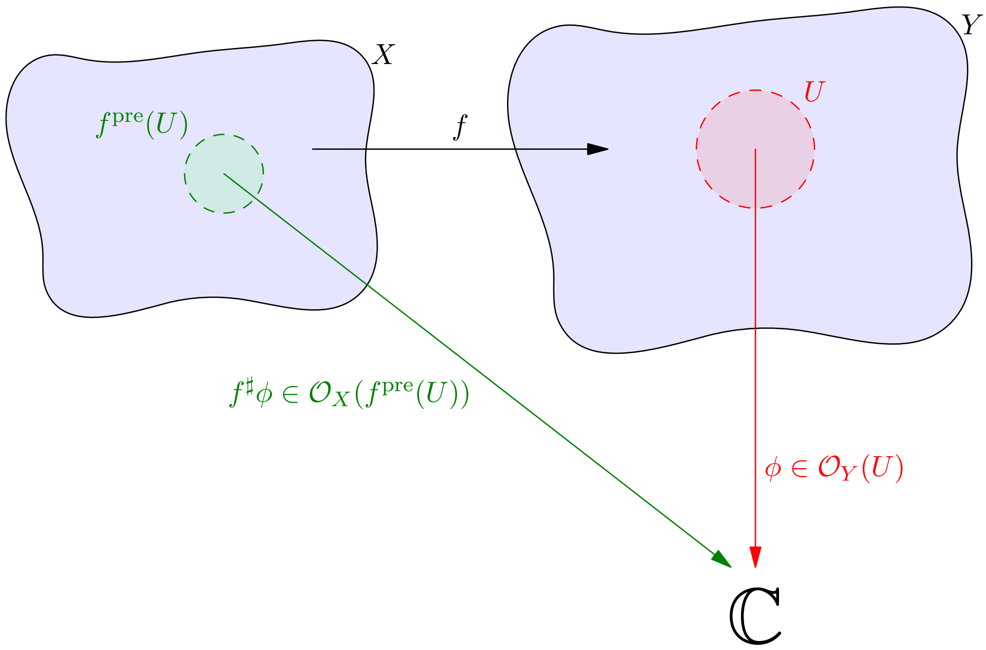
Definition 81.1.3. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be baby ringed spaces. A continuous map of topological spaces $f\colon X \to Y$ is a **morphism** if every pullback of a regular function on Y is a regular function on X.

Two baby ringed spaces are **isomorphic** if there are mutually inverse morphisms between them, which we then call **isomorphisms**.

In particular, the pullback gives us a (reversed) *ring homomorphism*

$$f^\sharp\colon \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{\text{pre}}(U))$$

for *every* U; thus our morphisms package a lot of information. Here's a picture of a morphism f, and the pullback of $\phi\colon U \to \mathbb{C}$ (where $U \subseteq Y$).



Example 81.1.4 (The pullback of $\frac{1}{y-25}$ under $t \mapsto t^2$)
The map

$$f\colon X = \mathbb{A}^1 \to Y = \mathbb{A}^1 \quad \text{by} \quad t \mapsto t^2$$

is a morphism of varieties. For example, consider the regular function $\varphi = \frac{1}{y-25}$ on the open set $Y \setminus \{25\} \subseteq Y$. The f-inverse image is $X \setminus \{\pm 5\}$. Thus the pullback is

$$f^\sharp\varphi\colon X \setminus \{\pm 5\} \to Y \setminus \{25\}$$
$$\text{by} \quad x \mapsto \frac{1}{x^2 - 25}$$

which is regular on $X \setminus \{\pm 5\}$.

§81.2 Classifying the simplest examples

Prototypical example for this section: Theorem 81.2.2; they're just polynomials.

On a philosophical point, we like the earlier definition because it adheres to our philosophy of treating our varieties as intrinsic objects, rather than embedded ones. However, it is somewhat of a nuisance to actually verify it.

So in this section, we will

- classify all the morphisms from $\mathbb{A}^m \to \mathbb{A}^n$, and
- classify all the morphisms from $\mathbb{CP}^m \to \mathbb{CP}^n$.

It what follows I will wave my hands a lot in claiming that something is a morphism, since doing so is mostly detail checking. The theorems which follow will give us alternative definitions of morphism which are more coordinate-based and easier to use for actual computations.

§81.2.i Affine classification

Earlier we saw how $t \mapsto t^2$ gives us a map. More generally, given any polynomial $P(t)$, the map $t \mapsto P(t)$ will work. And in fact, that's all:

> **Exercise 81.2.1.** Let $X = \mathbb{A}^1$, $Y = \mathbb{A}^1$. By considering $\mathrm{id} \in \mathcal{O}_Y(Y)$, show that no other regular functions exist.

In fact, let's generalize the previous exercise:

Theorem 81.2.2 (Regular maps of affine varieties are globally polynomials)
Let $X \subseteq \mathbb{A}^m$ and $Y \subseteq \mathbb{A}^n$ be affine varieties. Every morphism $f\colon X \to Y$ of varieties is given by

$$x = (x_1, \dots, x_m) \xmapsto{f} (P_1(x), \dots, P_n(x))$$

where $P_1, \dots, P_n$ are polynomials.

Proof. It's not too hard to see that all such functions work, so let's go the other way. Let $f\colon X \to Y$ be a morphism.

First, remark that $f^{\mathrm{pre}}(Y) = X$. Now consider the regular function $\pi_1 \in \mathcal{O}_Y(Y)$, given by the projection $(y_1, \dots, y_n) \mapsto y_1$. Thus we need $f \circ \pi_1$ to be regular on X.

But for affine varieties $\mathcal{O}_X(X)$ is just the coordinate ring $\mathbb{C}[X]$ and so we know there is a polynomial P_1 such that $f \circ \pi_1 = P_1$. Similarly for the other coordinates. $\square$

§81.2.ii Projective classification

Unfortunately, the situation is a little weirder in the projective setting. If $X \subseteq \mathbb{CP}^m$ and $Y \subseteq \mathbb{CP}^n$ are projective varieties, then every function

$$x = (x_0 : x_1 : \dots : x_m) \mapsto (P_0(x) : P_1(x) : \dots : P_n(x))$$

is a valid morphism, provided the P_i are homogeneous of the same degree and don't all vanish simultaneously. However if we try to repeat the proof for affine varieties we run into an issue: there is no π_1 morphism. (Would we send $(1:1) = (2:2)$ to 1 or 2?)

And unfortunately, there is no way to repair this. Counterexample:

Example 81.2.3 (Projective map which is not globally polynomial)
Let $V = \mathcal{V}_{\mathrm{pr}}(xy - z^2) \subseteq \mathbb{CP}^2$. Then the map

$$V \to \mathbb{CP}^1 \quad \text{by} \quad (x:y:z) \mapsto \begin{cases} (x:z) & x \neq 0 \\ (z:y) & y \neq 0 \end{cases}$$

turns out to be a morphism of projective varieties. This is well defined just because $(x : z) = (z : y)$ if $x, y \neq 0$; this should feel reminiscent of the definition of regular function.

The good news is that "local" issues are the only limiting factor.

Theorem 81.2.4 (Regular maps of projective varieties are locally polynomials)
Let $X \subseteq \mathbb{CP}^m$ and $Y \subseteq \mathbb{CP}^n$ be projective varieties and let $f \colon X \to Y$ be a morphism. Then at every point $p \in X$ there exists an open neighborhood $U_p \ni p$ and polynomials $P_0, P_1, \dots, P_n$ (which depend on U) so that

$$f(x) = (P_0(x) : P_1(x) : \dots : P_n(x)) \quad \forall x = (x_0 : \dots : x_n) \in U_p.$$

Of course the polynomials P_i must be homogeneous of the same degree and cannot vanish simultaneously on any point of U_p.

Example 81.2.5 (Example of an isomorphism)
In fact, the map $V = \mathcal{V}_{\text{pr}}(xy - z^2) \to \mathbb{CP}^1$ is an isomorphism. The inverse map $\mathbb{CP}^1 \to V$ is given by

$$(s : t) \mapsto (s^2 : t^2 : st).$$

Thus actually $V \cong \mathbb{CP}^1$.

§81.3 Some more applications and examples

Prototypical example for this section: $\mathbb{A}^1 \hookrightarrow \mathbb{CP}^1$ is a good one.

The previous section complete settles affine varieties to affine varieties, and projective varieties to projective varieties. However, the definition we gave at the start of the chapter works for *any* baby ringed spaces, and therefore there is still a lot of room to explore.

For example, **we can have affine spaces talk to projective ones**. Why not? The power of our pullback-based definition is that you enable any baby ringed spaces to communicate, even if they live in different places.

Example 81.3.1 (Embedding $\mathbb{A}^1 \hookrightarrow \mathbb{CP}^1$)
Consider a morphism

$$f \colon \mathbb{A}^1 \hookrightarrow \mathbb{CP}^1 \quad \text{by} \quad t \mapsto (t : 1).$$

This is also a morphism of varieties. (Can you see what the pullbacks look like?) This reflects the fact that $\mathbb{CP}^1$ is "$\mathbb{A}^1$ plus a point at infinity".

Here is another way you can generate more baby ringed spaces. Given any projective variety, you can take an open subset of it, and that will itself be a baby ringed space. We give this a name:

Definition 81.3.2. A **quasi-projective variety** is an open set X of a projective variety V. It is a baby ringed space $(X, \mathcal{O}_X)$ too, because for any open set $U \subseteq X$ we simply define $\mathcal{O}_X(U) = \mathcal{O}_V(U)$.

We chose to take open subsets of projective varieties because this will subsume the affine ones, for example:

Example 81.3.3 (The parabola is quasi-projective)
Consider the parabola $V = \mathcal{V}(y - x^2) \subset \mathbb{A}^2$. We take the projective variety $W = \mathcal{V}_{\text{pr}}(zy - x^2)$ and look at the standard affine chart $D(z)$. Then there is an isomorphism

$$\begin{aligned} V &\to D(z) \subseteq W \\ (x, y) &\mapsto (x : y : 1) \\ (x/z, y/z) &\leftarrow\!\shortmid (x : y : z). \end{aligned}$$

Consequently, V is (isomorphic to) an open subset of W, thus we regard it as quasi-projective.

In general this proof can be readily adapted:

Proposition 81.3.4 (Affine $\subseteq$ quasi-projective)
Every affine variety is isomorphic to a quasi-projective one (i.e. every affine variety is an open subset of a projective variety).

So quasi-projective varieties generalize both types of varieties we have seen.

§81.4 The hyperbola effect

Prototypical example for this section: $\mathbb{A}^1 \setminus \{0\}$ *is even affine*

So here is a natural question: are there quasi-projective varieties which are neither affine nor projective? The answer is yes, but for the sake of narrative I'm going to play dumb and find a *non-example*, with the actual example being given in the problems.

Our first guess might be to take the simplest projective variety, say $\mathbb{CP}^1$, and delete a point (to get an open set). This is quasi-projective, but it's isomorphic to $\mathbb{A}^1$. So instead we start with the simplest affine variety, say $\mathbb{A}^1$, and try to delete a point.

Surprisingly, this doesn't work.

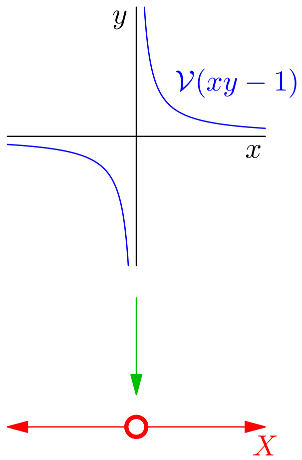
Example 81.4.1 (Crucial example: punctured line is isomorphic to hyperbola)
Let $X = \mathbb{A}^1 \setminus \{0\}$ be an quasi-projective variety. We claim that in fact we have an isomorphism

$$X \cong V = \mathcal{V}(xy - 1) \subseteq \mathbb{A}^2$$

which shows that X is still isomorphic to an affine variety. The maps are

$$\begin{aligned} X &\leftrightarrow V \\ t &\mapsto (t, 1/t) \\ x &\leftarrow\!\shortmid (x, y). \end{aligned}$$

Intuitively, the "hyperbola $y = 1/x$" in $\mathbb{A}^2$ can be projected onto the x-axis. Here is the relevant picture.



Actually, deleting any number of points from $\mathbb{A}^1$ fails. If we delete $\{1, 2, 3\}$, the resulting open set is isomorphic as a baby ringed space to $\mathcal{V}(y(x-1)(x-2)(x-3)-1)$, which colloquially might be called $y = \frac{1}{(x-1)(x-2)(x-3)}$.

The truth is more general.

Distinguished open sets of affine varieties are affine.

Here is the exact isomorphism.

Theorem 81.4.2 (Distinguished open subsets of affines are affine)

Consider $X = D(f) \subseteq V = \mathcal{V}(f_1, \dots, f_m) \subseteq \mathbb{A}^n$, where V is an affine variety, and the distinguished open set X is thought of as a quasi-projective variety. Define

$$W = \mathcal{V}(f_1, \dots, f_m, y \cdot f - 1) \subseteq \mathbb{A}^{n+1}$$

where y is the $(n+1)$st coordinate of $\mathbb{A}^{n+1}$.

Then $X \cong W$.

For lack of a better name, I will dub this the **hyperbola effect**, and it will play a significant role later on.

Therefore, if we wish to find an example of a quasi-projective variety which is not affine, one good place to look would be an open set of an affine space which is not distinguished open. If you are ambitious now, you can try to prove the punctured plane (that is, $\mathbb{A}^2$ minus the origin) works. We will see that example once again later in the next chapter, so you will have a second chance to do so.

§81.5 A few harder problems to think about

Problem 81A. Consider the map

$$\mathbb{A}^1 \to \mathcal{V}(y^2 - x^3) \subseteq \mathbb{A}^2 \quad \text{by} \quad t \mapsto (t^2, t^3).$$

Show that it is a morphism of varieties, but it is not an isomorphism.

Problem 81B†. Show that every projective variety has an open neighborhood which is isomorphic to an affine variety. In this way, "projective varieties are locally affine".

Problem 81C. Let V be a affine variety and let W be a irreducible projective variety. Prove that $V \cong W$ if and only if V and W are a single point.

Problem 81D (Punctured plane is not affine)**.** Let $X = \mathbb{A}^2 \setminus \{(0,0)\}$ be an open set of $\mathbb{A}^2$. Let V be any affine variety and let $f \colon X \to V$ be a morphism. Show that f is not an isomorphism.