XVIII

Algebraic Topology II: Homology

Part XVIII: Contents

71 Singular homology

Now that we've defined $\pi_1(X)$, we turn our attention to a second way of capturing the same idea, $H_1(X)$. We'll then define $H_n(X)$ for $n \geq 2$. The good thing about the H_n groups is that, unlike the π_n groups, they are much easier to compute in practice. The downside is that their definition will require quite a bit of setup, and the "algebraic" part of "algebraic topology" will become a lot more technical.

§71.1 Simplices and boundaries

Prototypical example for this section: $\partial[v_0, v_1, v_2] = [v_0, v_1] - [v_0, v_2] + [v_1, v_2]$.

First things first:

Definition 71.1.1. The **standard n-simplex**, denoted Δ^n, is defined as

$$\{(x_0, x_1, \dots x_n) \mid x_i \geq 0, x_0 + \dots + x_n = 1\}.$$

Hence it's the convex hull of some vertices $[v_0, \dots, v_n]$. Note that we keep track of the order $v_0, \dots, v_n$ of the vertices, for reasons that will soon become clear.

Given a topological space X, a **singular n-simplex** is a map $\sigma \colon \Delta^n \to X$.

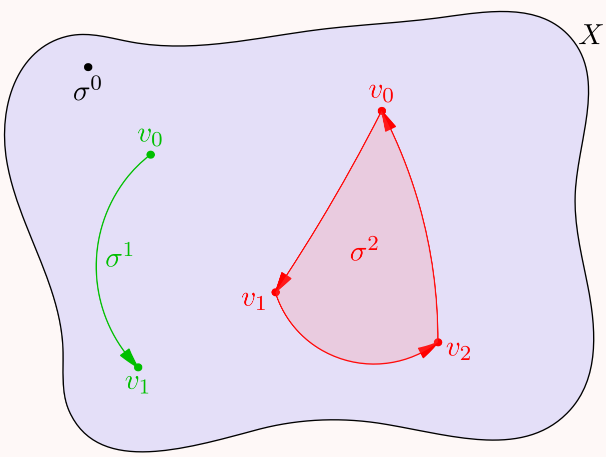
Example 71.1.2 (Singular simplices)

(a) Since $\Delta^0 = [v_0]$ is just a point, a singular 0-simplex X is just a point of X.

(b) Since $\Delta^1 = [v_0, v_1]$ is an interval, a singular 1-simplex X is just a path in X.

(c) Since $\Delta^2 = [v_0, v_1, v_2]$ is an equilateral triangle, a singular 2-simplex X looks a "disk" in X.

Here is a picture of all three in a space X:



The arrows aren't strictly necessary, but I've included them to help keep track of the "order" of the vertices; this will be useful in just a moment.

Now we're going to do something much like when we were talking about Stokes' theorem: we'll put a boundary ∂ operator on the singular n-simplices. This will give us a formal linear sums of n-simplices $\sum_k a_k\sigma_k$, which we call an **n-chain**.

In that case,

Definition 71.1.3. Given a singular n-simplex σ with vertices $[v_0, \dots, v_n]$, note that for every i we have an $(n-1)$ simplex $[v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n]$. The **boundary operator** ∂ is then defined by

$$\partial(\sigma) := \sum_i (-1)^i \left[v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n\right].$$

The boundary operator then extends linearly to n-chains:

$$\partial\left(\sum_k a_k\sigma_k\right) := \sum a_k \partial(\sigma_k).$$

By convention, a 0-chain has empty boundary.

Example 71.1.4 (Boundary operator)

Consider the chains depicted in Example 71.1.2. Then

(a) $\partial\sigma^0 = 0$.

(b) $\partial(\sigma^1) = [v_1] - [v_0]$: it's the "difference" of the 0-chain corresponding to point v_1 and the 0-chain corresponding to point v_0.

(c) $\partial(\sigma^2) = [v_0, v_1] - [v_0, v_2] + [v_1, v_2]$; i.e. one can think of it as the sum of the three oriented arrows which make up the "sides" of σ^2.

(d) Notice that if we take the boundary again, we get

$$\begin{aligned}\partial(\partial(\sigma^2)) &= \partial([v_0, v_1]) - \partial([v_0, v_2]) + \partial([v_1, v_2]) \\ &= ([v_1] - [v_0]) - ([v_2] - [v_0]) + ([v_2] - [v_1]) \\ &= 0.\end{aligned}$$

The fact that $\partial^2 = 0$ is of course not a coincidence.

Theorem 71.1.5 ($\partial^2 = 0$)

For any chain c, $\partial(\partial(c)) = 0$.

Proof. Essentially identical to Problem 45B: this is just a matter of writing down a bunch of $\sum$ signs. Diligent readers are welcome to try the computation. $\square$

Remark 71.1.6 — The eerie similarity between the chains used to integrate differential forms and the chains in homology is not a coincidence. The de Rham cohomology, discussed much later, will make the relation explicit.

§71.2 The singular homology groups

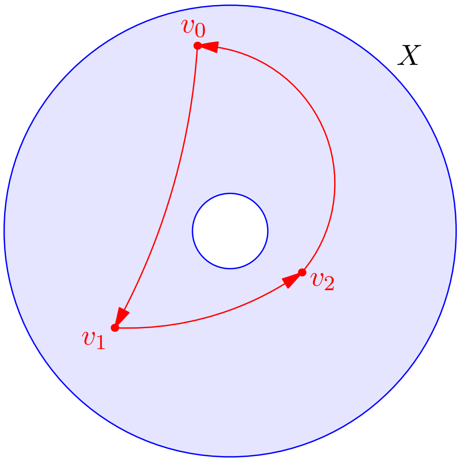
Prototypical example for this section: Probably $H_n(S^m)$, especially the case $m = n = 1$.

Let X be a topological space, and let $C_n(X)$ be the free abelian group of n-chains of X that we defined earlier. Our work above gives us a boundary operator ∂, so we have a sequence of maps

$$\dots \xrightarrow{\partial} C_3(X) \xrightarrow{\partial} C_2(X) \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\partial} 0$$

(here I'm using 0 to for the trivial group, which is standard notation for abelian groups.) We'll call this the **singular chain complex**.

Now, how does this let us detect holes in the space? To see why, let's consider an annulus, with a 1-chain c drawn in red:



Notice that

$$\partial c = ([v_1] - [v_0]) - ([v_2] - [v_0]) + ([v_2] - [v_1]) = 0$$

and so we can say this 1-chain c is a "cycle", because it has trivial boundary. However, c is not itself the boundary of any 2-chain, because of the hole in the center of the space — it's impossible to "fill in" the interior of c! So, we have detected the hole by the algebraic fact that

$$c \in \ker\left(C_1(X) \xrightarrow{\partial} C_0(X)\right) \qquad \text{but} \qquad c \notin \operatorname{im}\left(C_2(X) \xrightarrow{\partial} C_1(X)\right).$$

Indeed, if the hole was not present then this statement would be false.

Remark 71.2.1 — Note that homotopy and homology captures slightly different notion of "holes". For example, let T be a torus. Then, every map $S^2 \to T$ is nulhomotopic so $\pi_2(T)$ is trivial, but, as we will see in Proposition 72.3.6, $H_2(T) \cong \mathbb{Z}$. At least in the case of $n = 1$, then Theorem 71.2.7 states that for any path-connected space X and $x_0 \in X$, then $H_1(X)$ is the abelianization of $\pi_1(X, x_0)$, which is pretty much the best result you can expect — $H_1(X)$ must be abelian, while $\pi_1(X, x_0)$ need not be abelian. Nevertheless, it is still possible that $\pi_1(X, x_0)$ is nontrivial and $H_1(X)$ is trivial — see `https://math.stackexchange.com/q/1052414` for an example.

We can capture this idea in any dimension, as follows.

Definition 71.2.2. Let

$$\dots \xrightarrow{\partial} C_2(X) \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\partial} 0$$

as above. We say that $c \in C_n(X)$ is:

- a **cycle** if $c \in \ker\left(C_n(X) \xrightarrow{\partial} C_{n-1}(X)\right)$, and
- a **boundary** if $c \in \operatorname{im}\left(C_{n+1}(X) \xrightarrow{\partial} C_n(X)\right)$.

Denote the cycles and boundaries by $Z_n(X), B_n(X) \subseteq C_n(X)$, respectively.[1]

Question 71.2.3. Just to get you used to the notation: check that B_n and Z_n are themselves abelian groups, and that $B_n(X) \subseteq Z_n(X) \subseteq C_n(X)$.

The key point is that we can now define:

Definition 71.2.4. The **nth homology group** $H_n(X)$ is defined as

$$H_n(X) := Z_n(X)/B_n(X).$$

Example 71.2.5 (The zeroth homology group)

Let's compute $H_0(X)$ for a topological space X. We take $C_0(X)$, which is just formal linear sums of points of X.

First, we consider the kernel of $\partial \colon C_0(X) \to 0$, so the kernel of ∂ is the entire space $C_0(X)$: that is, every point is a "cycle".

Now, what is the boundary? The main idea is that $[b] - [a] = 0$ if and only if there's a 1-chain which connects a to b, i.e. there is a path from a to b. In particular,

$$X \text{ path connected} \implies H_0(X) \cong \mathbb{Z}.$$

More generally, we have

Proposition 71.2.6 (Homology groups split into path-connected components)

If $X = \bigcup_\alpha X_\alpha$ is a decomposition into path-connected components, then we have

$$H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha).$$

In particular, if X has r path-connected components, then $H_0(X) \cong \mathbb{Z}^{\oplus r}$.

(If it's surprising to see $\mathbb{Z}^{\oplus r}$, remember that an abelian group is the same thing as a $\mathbb{Z}$-module, so the notation $G \oplus H$ is customary in place of $G \times H$ when G, H are abelian.)

Now let's investigate the first homology group.

Theorem 71.2.7 (Hurewicz theorem)

Let X be path-connected. Then $H_1(X)$ is the *abelianization* of $\pi_1(X, x_0)$.

We won't prove this but you can see it roughly from the example. The group $H_1(X)$ captures the same information as $\pi_1(X, x_0)$: a cycle (in $Z_1(X)$) corresponds to the same thing as the loops we studied in $\pi_1(X, x_0)$, and the boundaries (in $B_1(X)$, i.e. the things we mod out by) are exactly the nulhomotopic loops in $\pi_1(X, x_0)$. The difference is that $H_1(X)$ allows loops to commute, whereas $\pi_1(X, x_0)$ does not.

[1] We don't use $C_n(X)$ to denote cycles — apart from the obvious reason that the notation is already used, the letter Z comes from the German word "Zyklus".

Remark 71.2.8 (Digression: category theory interpretation) — From this, you can say that there is a Hurewicz map $\pi_1(X, x_0) \xrightarrow{\phi} H_1(X)$ for each (X, x_0).
But there is more than that: this map is *natural*, in the sense that for $h \colon (X, x_0) \to (Y, y_0)$ map of pointed spaces, then

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "h_\sharp"] \arrow[d, "\phi"] & \pi_1(Y, y_0) \arrow[d, "\phi"] \\ H_1(X) \arrow[r, "h_*"] & H_1(Y) \end{tikzcd}$$

commutes.
In category theory terms, we say that ϕ is a *natural transformation* from π_1 to H_1.
Another way to say this is: we have families of groups

$$\{\pi_1(X, x_0) \mid (X, x_0) \text{ pointed space}\}$$

and

$$\{H_1(X) \mid (X, x_0) \text{ pointed space}\}$$

then the natural transformation ϕ can be seen as a family of homomorphisms

$$\{\phi \colon \pi_1(X, x_0) \to H_1(X) \mid (X, x_0) \text{ pointed space}\}$$

satisfying the naturality conditions.
Of course, the fact that π_1 is a functor means $\{\pi_1(X, x_0) \mid (X, x_0) \text{ pointed space}\}$ is a lot more than a family of groups indexed by pointed spaces, as explained in Theorem 65.6.2.

Example 71.2.9 (The first homology group of the annulus)

To give a concrete example, consider the annulus X above. We found a chain c that wrapped once around the hole of X. The point is that in fact,

$$H_1(X) = \langle c \rangle \cong \mathbb{Z}$$

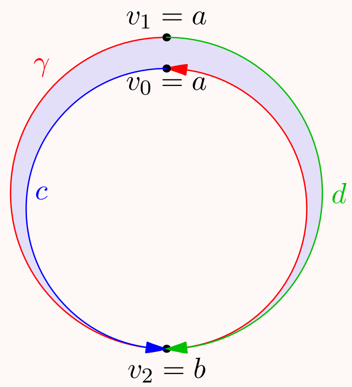
which is to say the chains c, $2c$, ... are all not the same in $H_1(X)$, but that any other 1-chain is equivalent to one of these. This captures the fact that X is really just S^1.

Example 71.2.10 (An explicit boundary in S^1)

In $X = S^1$, let a be the uppermost point and b the lowermost point. Let c be the simplex from a to b along the left half of the circle, and d the simplex from a to b along the right half. Finally, let γ be the simplex which represents a loop γ from a to itself, wrapping once counterclockwise around S^1. We claim that in $H^1(S^1)$ we have

$$\gamma = c - d$$

which geometrically means that $c - d$ represents wrapping once around the circle (which is of course what we expect).



Indeed this can be seen from the picture above, where we have drawn a 2-simplex whose boundary is exactly $\gamma - c + d$. The picture is somewhat metaphorical: in reality $v_0 = v_1 = a$, and the entire 2-simplex is embedded in S^1. This is why singular homology is so-called: the images of the simplex can sometimes look quite "singular".

Example 71.2.11 (The first homology group of the figure eight)

Consider X_8 (see Example 65.2.9). Both homology and homotopy see the two loops in X_8, call them a and b. The difference is that in $\pi_1(X_8, x_0)$, these two loops are not allowed to commute: we don't have $ab \neq ba$, because the group operation in π_1 is "concatenate paths". But in the homology group $H_1(X)$ the way we add a and b is to add them formally, to get the 1-chain $a + b$. So

$$H_1(X) \cong \mathbb{Z}^{\oplus 2} \quad \text{while} \quad \pi_1(X, x_0) = \langle a, b \rangle .$$

Example 71.2.12 (The homology groups of S^2)

Consider S^2, the two-dimensional sphere. Since it's path connected, we have $H_0(S^2) = \mathbb{Z}$. We also have $H_1(S^2) = 0$, for the same reason that $\pi_1(S^2)$ is trivial as well. On the other hand we claim that

$$H_2(S^2) \cong \mathbb{Z}.$$

The elements of $H_2(S^2)$ correspond to wrapping S^2 in a tetrahedral bag (or two bags, or three bags, etc.). Thus, the second homology group lets us detect the spherical cavity of S^2.[a]

[a] As remarked in Remark 71.2.1, unlike π_2, H_2 also detects other kinds of cavities, not just spherical.

Actually, more generally it turns out that we will have

$$H_n(S^m) \cong \begin{cases} \mathbb{Z} & n = m \text{ or } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Example 71.2.13 (Contractible spaces)
Given any contractible space X, it turns out that

$$H_n(X) \cong \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The reason is that, like homotopy groups, it turns out that homology groups are homotopy invariant. (We'll prove this next section.) So the homology groups of contractible X are the same as those of a one-point space, which are those above.

Example 71.2.14 (Homology groups of the torus)
While we won't be able to prove it for a while, it turns out that

$$H_n(S^1 \times S^1) \cong \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z}^{\oplus 2} & n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The homology group at 1 corresponds to our knowledge that $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$ and the homology group at 2 detects the "cavity" of the torus.

This is fantastic and all, but how does one go about actually computing any homology groups? This will be a rather long story, and we'll have to do a significant amount of both algebra and geometry before we're really able to compute any homology groups. In what follows, it will often be helpful to keep track of which things are purely algebraic (work for any chain complex), and which parts are actually stating something which is geometrically true.

§71.3 The homology functor and chain complexes

As I mentioned before, the homology groups are homotopy invariant. This will be a similar song and dance as the work we did to create a functor $\pi_1 \colon \mathsf{hTop}_* \to \mathsf{Grp}$. Rather than working slowly and pulling away the curtain to reveal the category theory at the end, we'll instead start with the category theory right from the start just to save some time.

Definition 71.3.1. The category hTop is defined as follows:

- Objects: topological spaces.
- Morphisms: *homotopy classes* of morphisms $X \to Y$.

In particular, X and Y are isomorphic in hTop if and only if they are homotopic.

You'll notice this is the same as hTop_*, except without the basepoints.

Theorem 71.3.2 (Homology is a functor hTop $\to$ Grp)

For any particular n, H_n is a functor $\mathsf{hTop} \to \mathsf{Grp}$. In particular,

- Given any map $f \colon X \to Y$, we get an induced map $f_* \colon H_n(X) \to H_n(Y)$.
- For two homotopic maps $f, g \colon X \to Y$, $f_* = g_*$.
- Two homotopic spaces X and Y have isomorphic homology groups: if $f \colon X \to Y$ is a homotopy then $f_* \colon H_n(X) \to H_n(Y)$ is an isomorphism.
- (Insert your favorite result about functors here.)

In order to do this, we have to describe how to take a map $f \colon X \to Y$ and obtain a map $H_n(f) \colon H_n(X) \to H_n(Y)$. Then we have to show that this map doesn't depend on the choice of homotopy. (This is the analog of the work we did with $f_\sharp$ before.) It turns out that this time around, proving this is much more tricky, and we will have to go back to the chain complex $C_\bullet(X)$ that we built at the beginning.

§71.3.i Algebra of chain complexes

Let's start with the algebra. First, I'll define the following abstraction of the complex to any sequence of abelian groups. Actually, though, it works in any category (not just AbGrp). The strategy is as follows: we'll define everything that we need completely abstractly, then show that the geometry concepts we want correspond to this setting.

Definition 71.3.3. A **chain complex** is a sequence of groups A_n and maps

$$\dots \xrightarrow{\partial} A_{n+1} \xrightarrow{\partial} A_n \xrightarrow{\partial} A_{n-1} \xrightarrow{\partial} \dots$$

such that the composition of any two adjacent maps is the zero morphism. We usually denote this by $A_\bullet$.

The nth homology group $H_n(A_\bullet)$ is defined as $\ker(A_n \to A_{n-1})/\operatorname{im}(A_{n+1} \to A_n)$. Cycles and boundaries are defined in the same way as before.

Obviously, this is just an algebraic generalization of the structure we previously looked at, rid of all its original geometric context.

Definition 71.3.4. A **morphism of chain complexes** (or chain map) $f \colon A_\bullet \to B_\bullet$ is a sequence of maps f_n for every n such that the diagram

$$\begin{tikzcd}
\dots \arrow[r, "\partial_A"] & A_{n+1} \arrow[r, "\partial_A"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "\partial_A"] \arrow[d, "f_n"] & A_{n-1} \arrow[r, "\partial_A"] \arrow[d, "f_{n-1}"] & \dots \\
\dots \arrow[r, "\partial_B"] & B_{n+1} \arrow[r, "\partial_B"] & B_n \arrow[r, "\partial_B"] & B_{n-1} \arrow[r, "\partial_B"] & \dots
\end{tikzcd}$$

commutes. Under this definition, the set of chain complexes becomes a category, which we denote Cmplx.

Note that given a morphism of chain complexes $f \colon A_\bullet \to B_\bullet$, every cycle in A_n gets sent to a cycle in B_n, since the square

$$\begin{tikzcd}
A_n \arrow[r, "\partial_A"] \arrow[d, "f_n"'] & A_{n-1} \arrow[d, "f_{n-1}"] \\
B_n \arrow[r, "\partial_B"'] & B_{n-1}
\end{tikzcd}$$

commutes. Similarly, every boundary in A_n gets sent to a boundary in B_n. Thus,

Every map of $f\colon A_\bullet \to B_\bullet$ gives a map $f_*\colon H_n(A) \to H_n(B)$ for every n.

Exercise 71.3.5. Interpret H_n as a functor $\mathsf{Cmplx} \to \mathsf{Grp}$.

Next, we want to define what it means for two maps f and g to be homotopic. Here's the answer:

Definition 71.3.6. Let $f, g\colon A_\bullet \to B_\bullet$. Suppose that one can find a map $P_n\colon A_n \to B_{n+1}$ for every n such that

$$g_n - f_n = \partial_B \circ P_n + P_{n-1} \circ \partial_A$$

Then P is a **chain homotopy** from f to g and f and g are **chain homotopic**.

We can draw a picture to illustrate this (warning: the diagonal dotted arrows do NOT commute with all the other arrows):

$$\begin{tikzcd}
\dots \arrow[r, "\partial_A"] & A_{n+1} \arrow[r, "\partial_A"] \arrow[d, "g-f"'] & A_n \arrow[r, "\partial_A"] \arrow[d, "g-f"'] \arrow[dl, dotted, "P_n"'] & A_{n-1} \arrow[r, "\partial_A"] \arrow[d, "g-f"'] \arrow[dl, dotted, "P_{n-1}"'] & \dots \\
\dots \arrow[r, "\partial_B"] & B_{n+1} \arrow[r, "\partial_B"] & B_n \arrow[r, "\partial_B"] & B_{n-1} \arrow[r, "\partial_B"] & \dots
\end{tikzcd}$$

The definition is that in each slanted "parallelogram", the $g - f$ arrow is the sum of the two compositions along the sides.

Remark 71.3.7 — This equation should look terribly unmotivated right now, aside from the fact that we are about to show it does the right algebraic thing. Its derivation comes from the geometric context that we have deferred until the next section, where "homotopy" will naturally give "chain homotopy".

Now, the point of this definition is that

Proposition 71.3.8 (Chain homotopic maps induce the same map on homology groups)
Let $f, g : A_\bullet \to B_\bullet$ be chain homotopic maps $A_\bullet \to B_\bullet$. Then the induced maps $f_*, g_*\colon H_n(A_\bullet) \to H_n(B_\bullet)$ coincide for each n.

Proof. It's equivalent to show $g - f$ gives the zero map on homology groups, In other words, we need to check that every cycle of A_n becomes a boundary of B_n under $g - f$.

Question 71.3.9. Verify that this is true. □

§71.3.ii Geometry of chain complexes

Now let's fill in the geometric details of the picture above. First:

Lemma 71.3.10 (Map of space $\implies$ map of singular chain complexes)
Each $f\colon X \to Y$ induces a map $C_n(X) \to C_n(Y)$.

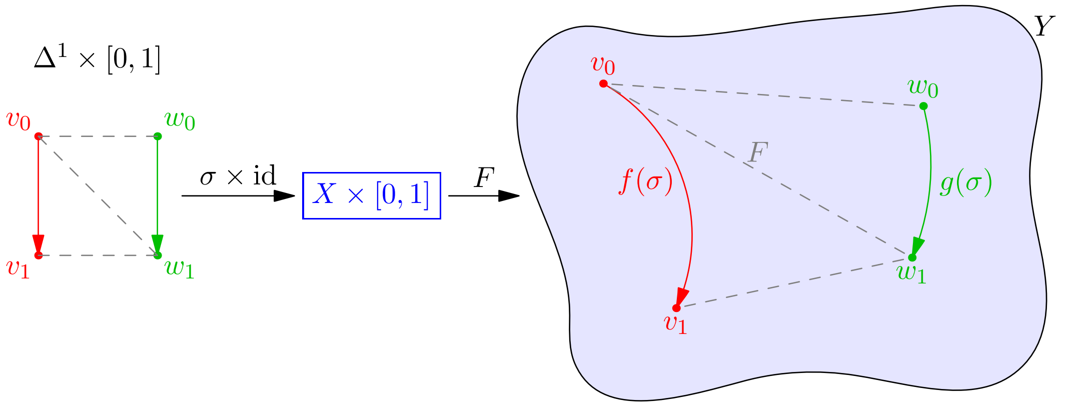
Proof. Take the composition

$$\Delta^n \xrightarrow{\sigma} X \xrightarrow{f} Y.$$

In other words, a path in X becomes a path in Y, et cetera. (It's not hard to see that the squares involving ∂ commute; check it if you like.) □

Now, what we need is to show that if $f, g \colon X \to Y$ are homotopic, then they are chain homotopic. To produce a chain homotopy, we need to take every n-simplex X to an $(n+1)$-chain in Y, thus defining the map P_n.

Let's think about how we might do this. Let's take the n-simplex $\sigma \colon \Delta^n \to X$ and feed it through f and g; pictured below is a 1-simplex σ (i.e. a path in X) which has been mapped into the space Y. Homotopy means the existence of a map $F \colon X \times [0,1] \to Y$ such that $F(-,0) = f$ and $F(-,1) = g$, parts of which I've illustrated below with grey arrows in the image for Y.



This picture suggests how we might proceed: we want to create a 2-chain on Y given the 1-chains we've drawn. The homotopy F provides us with a "square" structure on Y, i.e. the square bounded by v_0, v_1, w_1, w_0. We split this up into two triangles; and that's our 2-chain.

We can make this formal by taking $\Delta^1 \times [0,1]$ (which *is* a square) and splitting it into two triangles. Then, if we apply $\sigma \times \mathrm{id}$, we'll get an 2-chain in $X \times [0,1]$, and then finally applying F will map everything into our space Y. In our example, the final image is the 2-chain, consisting of two triangles, which in our picture can be written as $[v_0, w_0, w_1] - [v_0, v_1, w_1]$; the boundaries are given by the red, green, grey.

More generally, for an n-simplex $\phi = [x_0, \dots, x_n]$ we define the so-called *prism operator* P_n as follows. Set $v_i = f(x_i)$ and $w_i = g(x_i)$ for each i. Then, we let

$$P_n(\phi) \coloneqq \sum_{i=0}^{n} (-1)^i (F \circ (\phi \times \mathrm{id}))\, [v_0, \dots, v_i, w_i, \dots, w_n]\,.$$

This is just the generalization of the construction above to dimensions $n > 1$; we split $\Delta^n \times [0,1]$ into $n+1$ simplices, map it into X by $\phi \times \mathrm{id}$ and then push the whole thing into Y. The $(-1)^i$ makes sure that the "diagonal" faces all cancel off with each other.

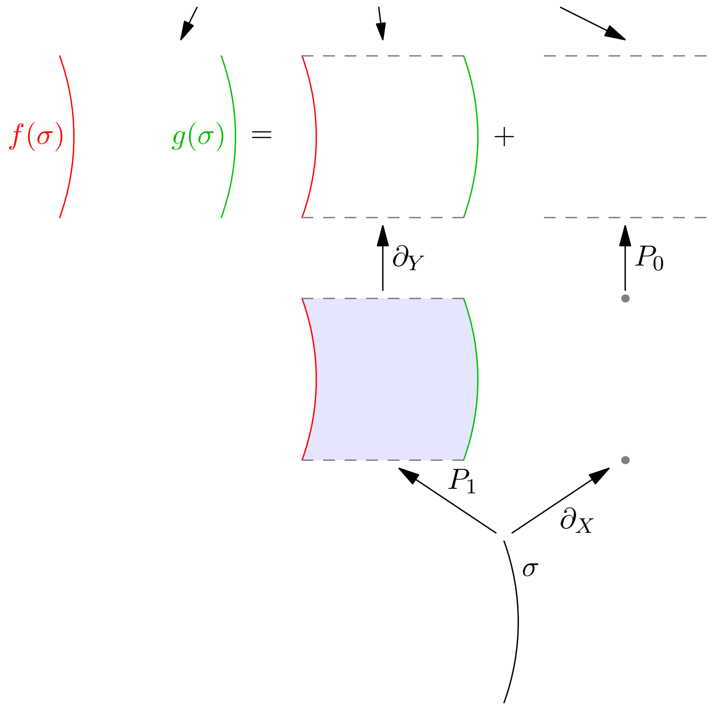
We now claim that for every σ,

$$\partial_Y(P_n(\sigma)) = g(\sigma) - f(\sigma) - P_{n-1}(\partial_X \sigma).$$

In the picture, $\partial_Y \circ P_n$ is the boundary of the entire prism (in the figure, this becomes the red, green, and grey lines, not including diagonal grey, which is cancelled out). The $g - f$ is the green minus the red, and the $P_{n-1} \circ \partial_X$ represents the grey edges of the prism (not including the diagonal line from v_1 to w_0). Indeed, one can check (just by writing down several $\sum$ signs) that the above identity holds.

As a picture:

$$\underbrace{g(\sigma) - f(\sigma)} = \underbrace{\partial_Y(P_n(\sigma))} + \underbrace{P_{n-1}(\partial_X(\sigma))}$$



So that gives the chain homotopy from f to g, completing the proof of Theorem 71.3.2.

§71.4 More examples of chain complexes

We now end this chapter by providing some more examples of chain complexes, which we'll use in the next chapter to finally compute topological homology groups.

Example 71.4.1 (Reduced homology groups)

Suppose X is a (nonempty) topological space. One can augment the standard singular complex as follows: do the same thing as before, but augment the end by adding a $\mathbb{Z}$, as shown:

$$\cdots \to C_1(X) \to C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

Here ε is defined by $\varepsilon(\sum n_i p_i) = \sum n_i$ for points $p_i \in X$. (Recall that a 0-chain is just a formal sum of points!) We denote this **augmented singular chain complex** by $\widetilde{C}_\bullet(X)$.

This may seem like a random thing to do, but it can be justified by taking the definitions we started with and "generalizing backwards". Recall that an n-simplex is given by $n+1$ vertices: $[v_0, \dots, v_n]$. That suggests that a (-1)-simplex is given by 0 vertices: $[]$!

We reach the same conclusion if we apply the definition of the standard n-simplex using $n = -1$. Δ^{-1} must be the subset of $\mathbb{R}^0$ consisting of all points whose coordinates are nonnegative and sum to 1. There are no such points, so $\Delta^{-1} = \{\}$.

Consequently, given a topological space X, a singular (-1)-simplex in X must be a function $\{\} \to X$. There is one such function: the *empty function*, whose image is the empty set.

That is, every topological space X has exactly one (-1)-simplex, which we identify with $\{\}$. Thus, the (-1)st chain group $C_{-1}(X)$ is the free abelian group generated by one element; ie, $\widetilde{C}_{-1}(X) \cong \mathbb{Z}$ (where the isomorphism identifies $\{\}$ with 1).

What about boundaries? To take the boundary of a simplex $[v_0, \dots, v_n]$, we remove each vertex one-by-one, and take the alternating sum. Therefore, $\partial([v]) = []$. Extending it linearly to complexes yields $\partial(\sum n_i p_i) = \sum n_i \cdot 1$ — so ε really is just the boundary operator, generalized to the case $\widetilde{C}_0(X) \xrightarrow{\partial} \widetilde{C}_{-1}(X)$.[a]

[a] What about $n \le -2$? An n-simplex comes from a list of vertices of length $(n+1)$, so a (-2)-simplex would require a list of vertices length (-1) — but there aren't any such lists. So while there is one (-1)-simplex, there are zero (-2)-simplices (ditto for $n < -2$). The free abelian group on zero elements is the trivial group, so $\widetilde{C}_{-2} \cong \mathbf{0}$. In particular, $\partial([]) = 0$.

Question 71.4.2. What's the homology of the above chain at $\mathbb{Z}$? (Hint: you need X nonempty.)

Definition 71.4.3. The homology groups of the augmented chain complex are called the **reduced homology groups** $\widetilde{H}_n(X)$ of the space X.

Obviously $\widetilde{H}_n(X) \cong H_n(X)$ for $n > 0$. But when $n = 0$, the map $H_0(X) \to \mathbb{Z}$ by ε has kernel $\widetilde{H}_0(X)$, thus $H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$.

This is usually just an added convenience. For example, it means that if X is contractible, then all its reduced homology groups vanish, and thus we won't have to keep fussing with the special $n = 0$ case.

Question 71.4.4. Given the claim earlier about $H_n(S^m)$, what should $\widetilde{H}_n(S^m)$ be?

Example 71.4.5 (Relative chain groups)

Suppose X is a topological space, and $A \subseteq X$ a subspace. We can "mod out" by A by defining

$$C_n(X, A) := C_n(X)/C_n(A)$$

for every n. Thus chains contained entirely in A are trivial.

Then, the usual ∂ on $C_n(X)$ generates a new chain complex

$$\dots \xrightarrow{\partial} C_{n+1}(X, A) \xrightarrow{\partial} C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \xrightarrow{\partial} \dots.$$

This is well-defined since ∂ takes $C_n(A)$ into $C_{n-1}(A)$.

Definition 71.4.6. The homology groups of the relative chain complex are the **relative homology groups** and denoted $H_n(X, A)$.

One naïve guess is that this might equal $H_n(X)/H_n(A)$. This is not true and in general doesn't even make sense; if we take X to be $\mathbb{R}^2$ and $A = S^1$ a circle inside it, we have $H_1(X) = H_1(\mathbb{R}^2) = 0$ and $H_1(S^1) = \mathbb{Z}$.

Another guess is that $H_n(X, A)$ might just be $\widetilde{H}_n(X/A)$. This will turn out to be true for most reasonable spaces X and A, and we will discuss this when we reach the excision theorem in Chapter 73.

Example 71.4.7 (Mayer-Vietoris sequence)

Suppose a space X is covered by two open sets U and V. We can define $C_n(U+V)$ as follows: it consists of chains such that each simplex is either entirely contained in U, or entirely contained in V.
Of course, ∂ then defines another chain complex

$$\dots \xrightarrow{\partial} C_{n+1}(U+V) \xrightarrow{\partial} C_n(U+V) \xrightarrow{\partial} C_{n-1}(U+V) \xrightarrow{\partial} \dots.$$

So once again, we can define homology groups for this complex; we denote them by $H_n(U+V)$. Miraculously, it will turn out that $H_n(U+V) \cong H_n(X)$.

§71.5 A few harder problems to think about

Problem 71A. For $n \ge 1$ show that the composition

$$S^{n-1} \hookrightarrow D^n \xrightarrow{F} S^{n-1}$$

cannot be the identity map on S^{n-1} for any continuous F.

Problem 71B (Brouwer fixed point theorem)**.** Use the previous problem to prove that any continuous function $f \colon D^n \to D^n$ has a fixed point.

72 The long exact sequence

In this chapter we introduce the key fact about chain complexes that will allow us to compute the homology groups of any space: the so-called "long exact sequence".

For those that haven't read about abelian categories: a sequence of morphisms of abelian groups

$$\cdots \to G_{n+1} \to G_n \to G_{n-1} \to \dots$$

is **exact** if the image of any arrow is equal to the kernel of the next arrow. In particular,

- The map $0 \to A \to B$ is exact if and only if $A \to B$ is injective.
- the map $A \to B \to 0$ is exact if and only if $A \to B$ is surjective.

(On that note: what do you call a chain complex whose homology groups are all trivial?)
A short exact sequence is one of the form $0 \to A \hookrightarrow B \twoheadrightarrow C \to 0$.

§72.1 Short exact sequences and four examples

Prototypical example for this section: Relative sequence and Mayer-Vietoris sequence.

Let $\mathcal{A} = \mathsf{AbGrp}$. Recall that we defined a morphism of chain complexes in $\mathcal{A}$ already.

Definition 72.1.1. Suppose we have a map of chain complexes

$$0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$$

It is said to be **short exact** if *each row* of the diagram below is short exact.

$$\begin{tikzcd}
& \vdots \arrow[d, "\partial_A"] & \vdots \arrow[d, "\partial_B"] & \vdots \arrow[d, "\partial_C"] & \\
0 \arrow[r] & A_{n+1} \arrow[r, hook, "f_{n+1}"] \arrow[d, "\partial_A"] & B_{n+1} \arrow[r, two heads, "g_{n+1}"] \arrow[d, "\partial_B"] & C_{n+1} \arrow[r] \arrow[d, "\partial_C"] & 0 \\
0 \arrow[r] & A_n \arrow[r, hook, "f_n"] \arrow[d, "\partial_A"] & B_n \arrow[r, two heads, "g_n"] \arrow[d, "\partial_B"] & C_n \arrow[r] \arrow[d, "\partial_C"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, hook, "f_{n-1}"] \arrow[d, "\partial_A"] & B_{n-1} \arrow[r, two heads, "g_{n-1}"] \arrow[d, "\partial_B"] & C_{n-1} \arrow[r] \arrow[d, "\partial_C"] & 0 \\
& \vdots & \vdots & \vdots &
\end{tikzcd}$$

This basically means $C_\bullet = B_\bullet / A_\bullet$, for suitable definition of / on chain complexes.

This agrees with the definition in Section 70.3.

Example 72.1.2 (Mayer-Vietoris short exact sequence and its augmentation)
Let $X = U \cup V$ be an open cover. For each n consider

$$\begin{array}{ccccc} C_n(U \cap V) & \hookrightarrow & C_n(U) \oplus C_n(V) & \twoheadrightarrow & C_n(U+V) \\ c & \longmapsto & (c, -c) & & \\ & & (c,d) & \longmapsto & c+d \end{array}$$

One can easily see (by taking a suitable basis) that the kernel of the latter map is exactly the image of the first map. This generates a short exact sequence

$$0 \to C_\bullet(U \cap V) \hookrightarrow C_\bullet(U) \oplus C_\bullet(V) \twoheadrightarrow C_\bullet(U+V) \to 0.$$

Example 72.1.3 (Augmented Mayer-Vietoris sequence)
We can *augment* each of the chain complexes in the Mayer-Vietoris sequence as well, by appending

$$\begin{tikzcd} 0 \arrow[r] & C_0(U \cap V) \arrow[r, hook] \arrow[d, "\varepsilon"'] & C_0(U) \oplus C_0(V) \arrow[r, two heads] \arrow[d, "\varepsilon \oplus \varepsilon"'] & C_0(U+V) \arrow[r] \arrow[d, "\varepsilon"'] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

to the bottom of the diagram. In other words we modify the above into

$$0 \to \widetilde{C}_\bullet(U \cap V) \hookrightarrow \widetilde{C}_\bullet(U) \oplus \widetilde{C}_\bullet(V) \twoheadrightarrow \widetilde{C}_\bullet(U+V) \to 0$$

where $\widetilde{C}_\bullet$ is the chain complex defined in Definition 71.4.3.

Example 72.1.4 (Relative chain short exact sequence)
Since $C_n(X, A) \coloneqq C_n(X)/C_n(A)$, we have a short exact sequence

$$0 \to C_\bullet(A) \hookrightarrow C_\bullet(X) \twoheadrightarrow C_\bullet(X, A) \to 0$$

for every space X and subspace A. This can be augmented: we get

$$0 \to \widetilde{C}_\bullet(A) \hookrightarrow \widetilde{C}_\bullet(X) \twoheadrightarrow C_\bullet(X, A) \to 0$$

by adding the final row

$$\begin{tikzcd} 0 \arrow[r] & C_0(A) \arrow[r, hook] \arrow[d, "\varepsilon"] & C_0(X) \arrow[r, two heads] \arrow[d, "\varepsilon"] & C_0(X, A) \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "\mathrm{id}"'] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0. \end{tikzcd}$$

§72.2 The long exact sequence of homology groups

Consider a short exact sequence $0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$. Now, we know that we get induced maps of homology groups, i.e. we have

$$\begin{tikzcd}
\vdots & \vdots & \vdots \\
H_{n+1}(A_\bullet) \arrow[r, "f_*"] & H_{n+1}(B_\bullet) \arrow[r, "g_*"] & H_{n+1}(C_\bullet) \\
H_n(A_\bullet) \arrow[r, "f_*"] & H_n(B_\bullet) \arrow[r, "g_*"] & H_n(C_\bullet) \\
H_{n-1}(A_\bullet) \arrow[r, "f_*"] & H_{n-1}(B_\bullet) \arrow[r, "g_*"] & H_{n-1}(C_\bullet) \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

But the theorem is that we can string these all together, taking each $H_{n+1}(C_\bullet)$ to $H_n(A_\bullet)$.

Theorem 72.2.1 (Short exact $\implies$ long exact)

Let $0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0$ be *any* short exact sequence of chain complexes we like. Then there is an *exact* sequence

$$\begin{tikzcd}
 & \dots \arrow[r] & H_{n+2}(C_\bullet) \arrow[dll, "\partial"'] \\
H_{n+1}(A_\bullet) \arrow[r, "f_*"'] & H_{n+1}(B_\bullet) \arrow[r, "g_*"] & H_{n+1}(C_\bullet) \arrow[dll, "\partial"'] \\
H_n(A_\bullet) \arrow[r, "f_*"'] & H_n(B_\bullet) \arrow[r, "g_*"] & H_n(C_\bullet) \arrow[dll, "\partial"'] \\
H_{n-1}(A_\bullet) \arrow[r, "f_*"'] & H_{n-1}(B_\bullet) \arrow[r, "g_*"] & H_{n-1}(C_\bullet) \arrow[dll, "\partial"'] \\
H_{n-2}(A_\bullet) \arrow[r] & \dots &
\end{tikzcd}$$

This is called a **long exact sequence** of homology groups.

Proof. A very long diagram chase, valid over any abelian category. (Alternatively, it's actually possible to use the snake lemma twice.) □

Remark 72.2.2 — The map $\partial \colon H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ can be written explicitly as follows. Recall that H_n is "cycles modulo boundaries", and consider the sub-diagram

$$\begin{tikzcd} & B_n \arrow[r, two heads, "g_n"] \arrow[d, "\partial_B"'] & C_n \arrow[d, "\partial_C"] \\ A_{n-1} \arrow[r, hook, "f_{n-1}"'] & B_{n-1} \arrow[r, two heads, "g_{n-1}"'] & C_{n-1} \end{tikzcd}$$

We need to take every cycle in C_n to a cycle in A_{n-1}. (Then we need to check a ton of "well-defined" issues, but let's put that aside for now.)
Suppose $c \in C_n$ is a cycle (so $\partial_C(c) = 0$). By surjectivity, there is a $b \in B_n$ with $g_n(b) = c$, which maps down to $\partial_B(b)$. Now, the image of $\partial_B(b)$ under g_{n-1} is zero by commutativity of the square, and so we can pull back under f_{n-1} to get a unique element of A_{n-1} (by exactness at B_{n-1}).
In summary: we go "*left, down, left*" to go from c to a:

$$\begin{tikzcd} & b \arrow[r, mapsto, "g_n"] \arrow[d, mapsto, "\partial_B"'] & \boxed{c} \arrow[d, mapsto, "\partial_C"] \\ \boxed{a} \arrow[r, mapsto, "f_{n-1}"'] & \partial_B(b) \arrow[r, mapsto, "g_{n-1}"'] & 0 \end{tikzcd}$$

Exercise 72.2.3. Check quickly that the recovered a is actually a cycle, meaning $\partial_A(a) = 0$. (You'll need another row, and the fact that $\partial_B^2 = 0$.)

The final word is that:

Short exact sequences of chain complexes give long exact sequences of homology groups.

In particular, let us take the four examples given earlier.

Example 72.2.4 (Mayer-Vietoris long exact sequence, provisional version)
The Mayer-Vietoris ones give, for $X = U \cup V$ an open cover,

$$\cdots \to H_n(U \cap V) \to H_n(U) \oplus H_n(V) \to H_n(U+V) \to H_{n-1}(U \cap V) \to \ldots .$$

and its reduced version

$$\cdots \to \widetilde{H}_n(U \cap V) \to \widetilde{H}_n(U) \oplus \widetilde{H}_n(V) \to \widetilde{H}_n(U+V) \to \widetilde{H}_{n-1}(U \cap V) \to \ldots .$$

This version is "provisional" because in the next section we will replace $H_n(U+V)$ and $\widetilde{H}_n(U+V)$ with something better. As for the relative homology sequences, we have:

Theorem 72.2.5 (Long exact sequence for relative homology)
Let X be a space, and let $A \subseteq X$ be a subspace. There are long exact sequences

$$\dots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \dots.$$

and

$$\dots \to \widetilde{H}_n(A) \to \widetilde{H}_n(X) \to H_n(X, A) \to \widetilde{H}_{n-1}(A) \to \dots.$$

The exactness of these sequences will give **tons of information** about $H_n(X)$ if only we knew something about what $H_n(U+V)$ or $H_n(X, A)$ looked like. This is the purpose of the next chapter.

§72.3 The Mayer-Vietoris sequence

Prototypical example for this section: The computation of $H_n(S^m)$ by splitting S^m into two hemispheres.

Now that we have done so much algebra, we need to invoke some geometry. There are two major geometric results in the Napkin. One is the excision theorem, which we discuss next chapter. The other we present here, which will let us take advantage of the Mayer-Vietoris sequence. The proofs are somewhat involved and are thus omitted; see [Ha02] for details.

The first theorem is that the notation $H_n(U+V)$ that we have kept until now is redundant, and can be replaced with just $H_n(X)$:

Theorem 72.3.1 (Open cover homology theorem)
Consider the inclusion $\iota \colon C_\bullet(U+V) \hookrightarrow C_\bullet(X)$. Then ι induces an isomorphism

$$H_n(U+V) \cong H_n(X).$$

Remark 72.3.2 — In fact, this is true for any open cover (even uncountable), not just those with two covers $U \cup V$. But we only state the special case with two open sets, because this is what is needed for Example 72.1.2.

So, Example 72.1.2 together with the above theorem implies, after replacing all the $H_n(U+V)$'s with $H_n(X)$'s:

Theorem 72.3.3 (Mayer-Vietoris long exact sequence)
If $X = U \cup V$ is an open cover, then we have long exact sequences

$$\dots \to H_n(U \cap V) \to H_n(U) \oplus H_n(V) \to H_n(X) \to H_{n-1}(U \cap V) \to \dots.$$

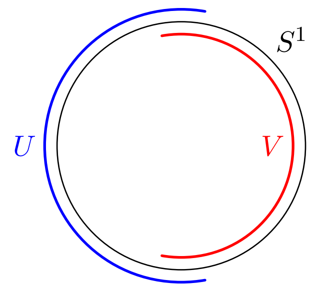
and

$$\dots \to \widetilde{H}_n(U \cap V) \to \widetilde{H}_n(U) \oplus \widetilde{H}_n(V) \to \widetilde{H}_n(X) \to \widetilde{H}_{n-1}(U \cap V) \to \dots.$$

At long last, we can compute the homology groups of the spheres.

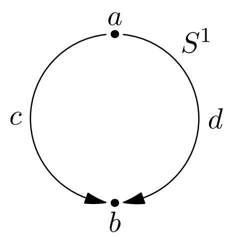
Theorem 72.3.4 (The homology groups of S^m)
For integers m and n,
$$\widetilde{H}_n(S^m) \cong \begin{cases} \mathbb{Z} & n = m \\ 0 & \text{otherwise.} \end{cases}$$
The generator $\widetilde{H}_n(S^n)$ is an n-cell which covers S^n exactly once (for example, the generator for $\widetilde{H}_1(S^1)$ is a loop which wraps around S^1 once).

Proof. This one's fun, so I'll only spoil the case $m = 1$, and leave the rest to you. Decompose the circle S^1 into two arcs U and V, as shown:



Each of U and V is contractible, so all their reduced homology groups vanish. Moreover, $U \cap V$ is homotopy equivalent to two points, hence
$$\widetilde{H}_n(U \cap V) \cong \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$
Now consider again the segment of the short exact sequence
$$\dots \to \underbrace{\widetilde{H}_n(U) \oplus \widetilde{H}_n(V)}_{=0} \to \widetilde{H}_n(S^1) \xrightarrow{\partial} \widetilde{H}_{n-1}(U \cap V) \to \underbrace{\widetilde{H}_{n-1}(U) \oplus \widetilde{H}_{n-1}(V)}_{=0} \to \dots.$$
From this we derive that $\widetilde{H}_n(S^1)$ is $\mathbb{Z}$ for $n = 1$ and 0 elsewhere.

It remains to analyze the generators of $\widetilde{H}_1(S^1)$. Note that the isomorphism was given by the connecting homomorphism ∂, which is given by a "left, down, left" procedure (Remark 72.2.2) in the diagram
$$\begin{tikzcd} & C_1(U) \oplus C_1(V) \arrow[r] \arrow[d, "\partial \oplus \partial"] & C_1(U+V) \\ C_0(U \cap V) \arrow[r] & C_0(U) \oplus C_0(V) & \end{tikzcd}$$
Mark the points a and b as shown in the two disjoint paths of $U \cap V$.



Then $a - b$ is a cycle which represents a generator of $H_0(U \cap V)$. We can find the pre-image of ∂ as follows: letting c and d be the chains joining a and b, with c contained in U, and d contained in V, the diagram completes as

$$\begin{tikzcd}
 & (c,d) \arrow[r, mapsto] \arrow[d, mapsto] & c-d \\
a-b \arrow[r, mapsto] & (a-b, a-b) &
\end{tikzcd}$$

In other words $\partial(c-d) = a-b$, so $c-d$ is a generator for $\widetilde{H}^1(S^1)$.

Thus we wish to show that $c-d$ is (in $H^1(S^1)$) equivalent to the loop γ wrapping around S^1 once, counterclockwise. This was illustrated in Example 71.2.10. $\square$

Thus, the key idea in Mayer-Vietoris is that

Mayer-Vietoris lets us compute $H_n(X)$ by splitting X into two open sets.

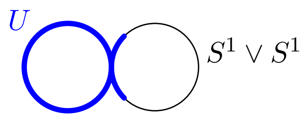
Here are some more examples.

Proposition 72.3.5 (The homology groups of the figure eight)
Let $X = S^1 \vee S^1$ be the figure eight. Then

$$\widetilde{H}_n(X) \cong \begin{cases} \mathbb{Z}^{\oplus 2} & n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The generators for $\widetilde{H}_1(X)$ are the two loops of the figure eight.

Proof. Again, for simplicity we work with reduced homology groups. Let U be the "left" half of the figure eight plus a little bit of the right, as shown below.



The set V is defined symmetrically. In this case $U \cap V$ is contractible, while each of U and V is homotopic to S^1.

Thus, we can read a segment of the long exact sequence as

$$\dots \to \underbrace{\widetilde{H}_n(U \cap V)}_{=0} \to \widetilde{H}_n(U) \oplus \widetilde{H}_n(V) \to \widetilde{H}_n(X) \to \underbrace{\widetilde{H}_{n-1}(U \cap V)}_{=0} \to \dots.$$

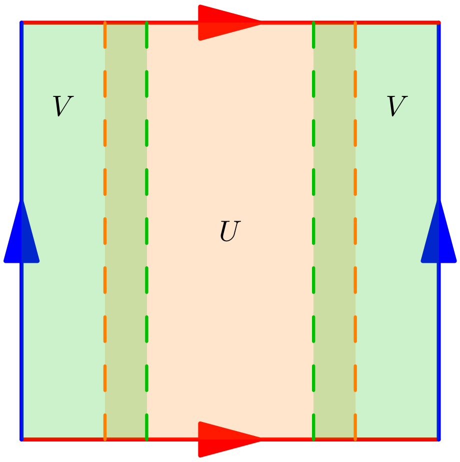
So we get that $\widetilde{H}_n(X) \cong \widetilde{H}_n(S^1) \oplus \widetilde{H}_n(S^1)$, The claim about the generators follows from the fact that, according to the isomorphism above, the generators of $\widetilde{H}_n(X)$ are the generators of $\widetilde{H}_n(U)$ and $\widetilde{H}_n(V)$, which we described geometrically in the last theorem. $\square$

Up until now, we have been very fortunate that we have always been able to make certain parts of the space contractible. This is not always the case, and in the next example we will have to actually understand the maps in question to complete the solution.

Proposition 72.3.6 (Homology groups of the torus)
Let $X = S^1 \times S^1$ be the torus. Then

$$\widetilde{H}_n(X) = \begin{cases} \mathbb{Z}^{\oplus 2} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. To make our diagram look good on 2D paper, we'll represent the torus as a square with its edges identified, though three-dimensionally the picture makes sense as well. Consider U (shaded light orange) and V (shaded green) as shown. (Note that V is connected due to the identification of the left and right (blue) edges, even if it doesn't look connected in the picture).



In the three dimensional picture, U and V are two cylinders which together give the torus. This time, U and V are each homotopic to S^1, and the intersection $U \cap V$ is the disjoint union of two circles: thus $\widetilde{H}_1(U \cap V) \cong \mathbb{Z} \oplus \mathbb{Z}$, and $H_0(U \cap V) \cong \mathbb{Z}^{\oplus 2} \implies \widetilde{H}_0(U \cap V) \cong \mathbb{Z}$.

For $n \geq 3$, we have

$$\dots \to \underbrace{\widetilde{H}_n(U \cap V)}_{=0} \to \widetilde{H}_n(U) \oplus \widetilde{H}_n(V) \to \widetilde{H}_n(X) \to \underbrace{\widetilde{H}_{n-1}(U \cap V)}_{=0} \to \dots .$$

and so $H_n(X) \cong 0$ for $n \geq 3$. Also, we have $H_0(X) \cong \mathbb{Z}$ since X is path-connected. So it remains to compute $H_2(X)$ and $H_1(X)$.

Let's find $H_2(X)$ first. We first consider the segment

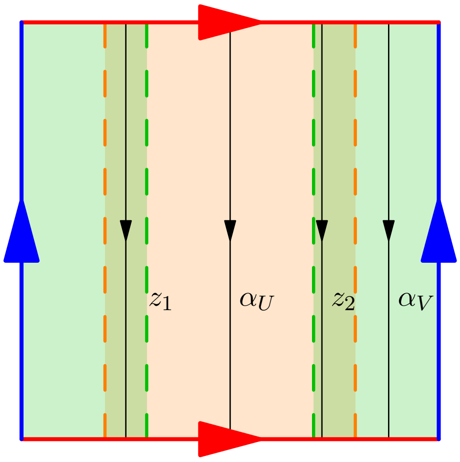
$$\dots \to \underbrace{\widetilde{H}_2(U) \oplus \widetilde{H}_2(V)}_{=0} \to \widetilde{H}_2(X) \overset{\partial}{\hookrightarrow} \underbrace{\widetilde{H}_1(U \cap V)}_{\cong \mathbb{Z} \oplus \mathbb{Z}} \xrightarrow{\phi} \underbrace{\widetilde{H}_1(U) \oplus \widetilde{H}_1(V)}_{\cong \mathbb{Z} \oplus \mathbb{Z}} \to \dots$$

Unfortunately, this time it's not immediately clear what $\widetilde{H}_2(X)$ because we only have one zero at the left. In order to do this, we have to actually figure out what the maps ∂ and ϕ look like. Note that, as we'll see, ϕ isn't an isomorphism even though the groups are isomorphic.

The presence of the zero term has allowed us to make the connecting map ∂ injective. First, $\widetilde{H}_2(X)$ is isomorphic to the image of ∂, which is exactly the kernel of the arrow ϕ inserted. To figure out what $\ker \phi$ is, we have to think back to how the map $C_\bullet(U \cap V) \to C_\bullet(U) \oplus C_\bullet(V)$ was constructed: it was $c \mapsto (c, -c)$. So the induced maps of homology groups is actually what you would guess: a 1-cycle z in $\widetilde{H}_1(U \cap V)$ gets sent $(z, -z)$ in $\widetilde{H}_1(U) \oplus \widetilde{H}_1(V)$.

In particular, consider the two generators z_1 and z_2 of $\widetilde{H}_1(U \cap V) = \mathbb{Z} \oplus \mathbb{Z}$, i.e. one cycle in each connected component of $U \cap V$. (To clarify: $U \cap V$ consists of two "wristbands"; z_i wraps around the ith one once.) Moreover, let α_U denote a generator of $\widetilde{H}_1(U) \cong \mathbb{Z}$, and α_V a generator of $\widetilde{H}_1(V) \cong \mathbb{Z}$.

The elements are depicted below:



Note that z_1, z_2, α_U, α_V are elements of the homology group, so you can move the paths around a bit — for instance, as elements of $\widetilde{H}_1(U)$, the chain drawn as z_1 and α_U represents the same element.

Then we have that

$$z_1 \mapsto (\alpha_U, -\alpha_V) \qquad \text{and} \qquad z_2 \mapsto (\alpha_U, -\alpha_V).$$

(The signs may differ on which direction you pick for the generators; note that $\mathbb{Z}$ has two possible generators.) We can even format this as a matrix:

$$\phi = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}.$$

And we observe $\phi(z_1 - z_2) = 0$, meaning this map has nontrivial kernel! That is,

$$\ker \phi = \langle z_1 - z_2 \rangle \cong \mathbb{Z}.$$

Thus, $\widetilde{H}_2(X) \cong \operatorname{im} \partial \cong \ker \phi \cong \mathbb{Z}$. We'll also note that $\operatorname{im} \phi$ is the set generated by $(\alpha_U, -\alpha_V)$; (in particular $\operatorname{im} \phi \cong \mathbb{Z}$ and the quotient by $\operatorname{im} \phi$ is $\mathbb{Z}$ too).

The situation is similar with $\widetilde{H}_1(X)$: this time, we have

$$\dots \xrightarrow{\phi} \underbrace{\widetilde{H}_1(U) \oplus \widetilde{H}_1(V)}_{\cong \mathbb{Z} \oplus \mathbb{Z}} \xrightarrow{\psi} \widetilde{H}_1(X) \overset{\partial}{\twoheadrightarrow} \underbrace{\widetilde{H}_0(U \cap V)}_{\cong \mathbb{Z}} \to \underbrace{\widetilde{H}_0(U) \oplus \widetilde{H}_0(V)}_{=0} \to \dots$$

and so we know that the connecting map ∂ is surjective, hence $\operatorname{im} \partial \cong \mathbb{Z}$. Now, we also have

$$\begin{aligned} \ker \partial \cong \operatorname{im} \psi &\cong \left(\widetilde{H}_1(U) \oplus \widetilde{H}_1(V) \right) / \ker \psi \\ &\cong \left(\widetilde{H}_1(U) \oplus \widetilde{H}_1(V) \right) / \operatorname{im} \phi \cong \mathbb{Z} \end{aligned}$$

by what we knew about $\operatorname{im}\phi$ already. To finish off we need some algebraic tricks. The first is Proposition 70.5.1, which gives us a short exact sequence

$$0 \to \underbrace{\ker \partial}_{\cong \operatorname{im}\psi \cong \mathbb{Z}} \hookrightarrow \widetilde{H}_1(X) \twoheadrightarrow \underbrace{\operatorname{im}\partial}_{\cong \mathbb{Z}} \to 0.$$

You should satisfy yourself that $\widetilde{H}_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ is the only possibility, but we'll prove this rigorously with Lemma 72.3.8. □

Remark 72.3.7 — Earlier, we remarked (without proof) that $\pi_2(X)$ is trivial — that is, homotopy does not found any "2-dimensional holes" in the torus. Why is it that $H_2(X) \cong \mathbb{Z}$?

You may want to manually compute the nontrivial element in $H_2(X)$ using the long exact sequence using the following method. Look at the long exact sequence:

$$\begin{tikzcd} \cdots \arrow[r] & \underbrace{H_2(U) \oplus H_2(V)}_{=0} \arrow[r] & \underbrace{H_2(X)}_{\cong\mathbb{Z}} \arrow[dll, "\partial"] \\ \underbrace{H_1(U \cap V)}_{\cong \mathbb{Z}\oplus\mathbb{Z}} \arrow[r, "\phi"'] & \underbrace{H_1(U) \oplus H_1(V)}_{\cong \mathbb{Z}\oplus\mathbb{Z}} \arrow[r] & \cdots \end{tikzcd}$$

We wish to find some nontrivial element in $H_2(X)$ — in order to do that, we can take an element in $\ker\phi \subseteq H_1(U \cap V)$ and take its preimage under ∂.

For that, $z_1 - z_2$ would suffice. In order to take its preimage under ∂, we need to recall how ∂ was constructed — it was a "left, down, left" procedure in the diagram:

$$\begin{tikzcd} & C_2(U) \oplus C_2(V) \arrow[r, two heads] \arrow[d] & C_2(X) \\ C_1(U \cap V) \arrow[r, hook] & C_1(U) \oplus C_1(V) & \end{tikzcd}$$

So, we find a (closed) element in $C_1(U \cap V)$ whose image under the quotient map is $z_1 - z_2$, then move it "right, up, right" to an element in $C_2(X)$.

If you did everything correctly, the result should be *the whole torus*!

Which emphasizes the point:

A "hole" detected by homology need not look like the interior of S^n.

Note that the previous example is of a different attitude than the previous ones, because we had to figure out what the maps in the long exact sequence actually were to even compute the groups. In principle, you could also figure out all the isomorphisms in the previous proof and explicitly compute the generators of $\widetilde{H}_1(S^1 \times S^1)$, but to avoid getting bogged down in detail I won't do so here.

Finally, to fully justify the last step, we present:

Lemma 72.3.8 (Splitting lemma)

For a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of abelian groups, the following are equivalent:

(a) There exists $p \colon B \to A$ such that $A \xrightarrow{f} B \xrightarrow{p} A$ is the identity.

(b) There exists $s \colon C \to B$ such that $C \xrightarrow{s} B \xrightarrow{g} C$ is the identity.

(c) There is an isomorphism from B to $A \oplus C$ such that the diagram

$$\begin{tikzcd}
& & B \arrow[dd, leftrightarrow, "\cong"] \arrow[dr, "g"] & & \\
0 \arrow[r] & A \arrow[ur, hook, "f"] \arrow[dr, hook] & & C \arrow[r] & 0 \\
& & A \oplus C \arrow[ur] & &
\end{tikzcd}$$

commutes. (The maps attached to $A \oplus C$ are the obvious ones.)

In particular, (b) holds anytime C is free.

In these cases we say the short exact sequence **splits**. The point is that

An exact sequence which splits let us obtain B given A and C.

In particular, for $C = \mathbb{Z}$ or any free abelian group, condition (b) is necessarily true. So, once we obtained the short exact sequence $0 \to \mathbb{Z} \to \widetilde{H}_1(X) \to \mathbb{Z} \to 0$, we were done.

Remark 72.3.9 — Unfortunately, not all exact sequences split: An example of a short exact sequence which doesn't split is

$$0 \to \mathbb{Z}/2\mathbb{Z} \xhookrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z} \to 0$$

since it is not true that $\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Remark 72.3.10 — The splitting lemma is true in any abelian category. The "direct sum" is the colimit of the two objects A and C.

§72.4 A few harder problems to think about

Problem 72A. Complete the proof of Theorem 72.3.4, i.e. compute $H_n(S^m)$ for all m and n. (Try doing $m = 2$ first, and you'll see how to proceed.)

Problem 72B. Compute the reduced homology groups of $\mathbb{R}^n$ with $p \geq 1$ points removed.

Problem 72C*. Let $n \geq 1$ and $k \geq 0$ be integers. Compute $H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$.

Problem 72D (Nine lemma)**.** Consider a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_2 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_3 \arrow[r] \arrow[d] & B_3 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and assume that all rows are exact, and two of the columns are exact. Show that the third column is exact as well.

Problem 72E* (Klein bottle)**.** Show that the reduced homology groups of the Klein bottle K are given by

$$\widetilde{H}_n(K) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 72F* (Triple long exact sequence)**.** Let $A \subseteq B \subseteq X$ be subspaces. Show that there is a long exact sequence

$$\cdots \to H_n(B, A) \to H_n(X, A) \to H_n(X, B) \to H_{n-1}(B, A) \to \dots.$$

73 Excision and relative homology

We have already seen how to use the Mayer-Vietoris sequence: we started with a sequence

$$\cdots \to H_n(U \cap V) \to H_n(U) \oplus H_n(V) \to H_n(U+V) \to H_{n-1}(U \cap V) \to \dots$$

and its reduced version, then appealed to the geometric fact that $H_n(U+V) \cong H_n(X)$. This allowed us to algebraically make computations on $H_n(X)$.

In this chapter, we turn our attention to the long exact sequence associated to the chain complex

$$0 \to C_n(A) \hookrightarrow C_n(X) \twoheadrightarrow C_n(X, A) \to 0.$$

The setup will look a lot like the previous two chapters, except in addition to $H_n \colon \mathsf{hTop} \to \mathsf{Grp}$ we will have a functor $H_n \colon \mathsf{hPairTop} \to \mathsf{Grp}$ which takes a pair (X, A) to $H_n(X, A)$. Then, we state (again without proof) the key geometric result, and use this to make deductions.

§73.1 Motivation

The main motivation is that:

Relative homology is the algebraic analog of quotient space.

So, for instance, when you see a map of pairs $f \colon (X, A) \to (Y, B)$, you should think of $X/A \to Y/B$.

Which explains the "reasonable guess" that for spaces $A \subseteq X$, we have $H_n(X, A) \cong \widetilde{H}_n(X/A)$.

By Theorem 73.4.3, the guess above is indeed true for most spaces. For example:

> **Question 73.1.1.** Let $X = [0, 1]$ and $A = \{0, 1\}$. Show that $H_1(X/A)$ and $H_1(X, A)$ are isomorphic to $\mathbb{Z}$. (In this example, so is $\pi_1(X/A)$.)

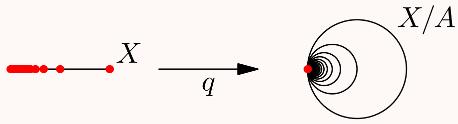
But not all. Similar to Example 64.2.6, if A is not closed, weird things can happen:

Example 73.1.2 ($H_n(X, A)$ where A is open in X)
Let $X = D^2$ be the closed disk.
If A is reasonably nice, for instance $A = S^1$ the boundary of X, we have $H_2(X, A) \cong H_2(X/A) \cong \mathbb{Z}$.
However, if $A = X \setminus \{0\}$ where 0 is the center of X, then $H_2(X, A)$ is still isomorphic to $\mathbb{Z}$; however $H_2(X/A) \cong 0$. (The latter isomorphism is harder to see, mainly because X/A is a weird space — it's not Hausdorff.)

Even when A is closed in X, problems can still happen.

Example 73.1.3 (The shrinking wedge of circles)

Let X be the interval $[0,1]$, and $A \subseteq X$ be $A = \{\frac{1}{n} \mid n \in \mathbb{Z}^+\} \cup \{0\}$. In this case, the quotient X/A would be isomorphic to the shrinking wedge of circles, as depicted below.



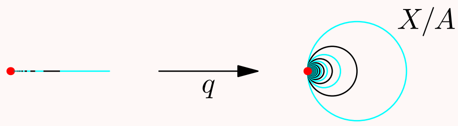
Note that in X/A, any open neighborhood of the red dot A/A must contains all but finitely many circles.

We claim that:

$$H_1(X, A) \ncong \widetilde{H}_1(X/A).$$

What could go wrong? Generally speaking, when you work algebraically then everything are finite, while in topology you have to consider things related to infinity.

Consider the following 1-simplex in $C(X/A)$, depicted in cyan.



Every element of $H(X, A)$ has a representative in $C(X)$ as a 1-cycles, which comprises of finitely many 1-simplices, each 1-simplex is equivalent to a segment $[a, b]$ — modulo a difference of a 1-boundary. Thus, intuitively, every element of $H(X, A)$ can only cover "finitely many circles" (or all but finitely many).

We haven't had enough tools to formalize all these yet. Formally speaking, the quotient maps $q\colon X \to X/A$ and $q\colon A \to A/A$ induces $q_*\colon H_1(X, A) \to H_1(X/A, A/A)$, and q_* is not injective.

Regardless, for nice spaces $A \subseteq X$ such that $H_n(X, A) \cong \widetilde{H}_n(X/A)$, we would be able to compute $H_n(X)$ based on $H_n(A)$ and $\widetilde{H}_n(X/A)$ — note that A and X/A is, in some sense, smaller and simpler than X.

§73.2 The long exact sequences

Recall Theorem 72.2.5, which says that the sequences

$$\dots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \dots.$$

and

$$\dots \to \widetilde{H}_n(A) \to \widetilde{H}_n(X) \to H_n(X, A) \to \widetilde{H}_{n-1}(A) \to \dots$$

are long exact. By Problem 72F* we even have a long exact sequence

$$\dots \to H_n(B, A) \to H_n(X, A) \to H_n(X, B) \to H_{n-1}(B, A) \to \dots.$$

for $A \subseteq B \subseteq X$.

This is the analog of the fact that X/B is homeomorphic to $\frac{X/A}{B/A}$ — we "cancel the common factor in the fraction".

An application of the first long exact sequence above gives:

Lemma 73.2.1 (Homology relative to contractible spaces)
Let X be a topological space, and let $A \subseteq X$ be contractible. For all n,

$$H_n(X, A) \cong \widetilde{H}_n(X).$$

Proof. Since A is contractible, we have $\widetilde{H}_n(A) = 0$ for every n. For each n there's a segment of the long exact sequence given by

$$\dots \to \underbrace{\widetilde{H}_n(A)}_{=0} \to \widetilde{H}_n(X) \to H_n(X, A) \to \underbrace{\widetilde{H}_{n-1}(A)}_{=0} \to \dots.$$

So since $0 \to \widetilde{H}_n(X) \to H_n(X, A) \to 0$ is exact, this means $H_n(X, A) \cong \widetilde{H}_n(X)$. $\square$

In particular, the theorem applies if A is a single point. The case $A = \varnothing$ is also worth noting. We compile these results into a lemma:

Lemma 73.2.2 (Relative homology generalizes absolute homology)
Let X be any space, and $* \in X$ a point. Then for all n,

$$H_n(X, \{*\}) \cong \widetilde{H}_n(X) \qquad \text{and} \qquad H_n(X, \varnothing) = H_n(X).$$

§73.3 The category of pairs

Since we now have an $H_n(X, A)$ instead of just $H_n(X)$, a natural next step is to create a suitable category of *pairs* and give ourselves the same functorial setup as before.

Definition 73.3.1. Let $\varnothing \neq A \subseteq X$ and $\varnothing \neq B \subseteq Y$ be subspaces, and consider a map $f\colon X \to Y$. If $f^{\text{img}}(A) \subseteq B$ we write

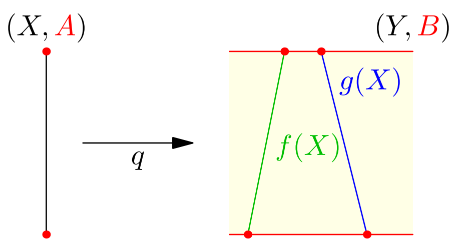
$$f\colon (X, A) \to (Y, B).$$

We say f is a **map of pairs**, between the pairs (X, A) and (Y, B).

Definition 73.3.2. We say that $f, g\colon (X, A) \to (Y, B)$ are **pair-homotopic** if they are "homotopic through maps of pairs".

More formally, a **pair-homotopy** $f, g\colon (X, A) \to (Y, B)$ is a map $F\colon [0, 1] \times X \to Y$, which we'll write as $F_t(X)$, such that F is a homotopy of the maps $f, g\colon X \to Y$ and each F_t is itself a map of pairs.

A typical $f, g\colon (X, A) \to (Y, B)$ that are pair-homotopic might look like this. Note that for all $t \in [0, 1]$, we must have $F_t^{\text{img}}(A) \subseteq B$.



Thus, we naturally arrive at two categories:

- PairTop, the category of *pairs* of topological spaces, and
- hPairTop, the same category except with maps only equivalent up to homotopy.

Definition 73.3.3. As before, we say pairs (X, A) and (Y, B) are **pair-homotopy equivalent** if they are isomorphic in hPairTop. An isomorphism of hPairTop is a **pair-homotopy equivalence**.

Remark 73.3.4 — Pair-homotopy equivalence of pairs is the natural generalization of homotopy equivalence of spaces, as defined in Definition 65.5.3. In fact, if $A = B = \varnothing$ then we have X is homotopy equivalent to Y if and only if $(X, \varnothing)$ is pair-homotopy equivalent to $(Y, \varnothing)$.

We can do the same song and dance as before with the prism operator to obtain:

Lemma 73.3.5 (Induced maps of relative homology)
We have a functor

$$H_n \colon \mathsf{hPairTop} \to \mathsf{Grp}.$$

That is, if $f \colon (X, A) \to (Y, B)$ then we obtain an induced map

$$f_* \colon H_n(X, A) \to H_n(Y, B).$$

and if two such f and g are pair-homotopic then $f_* = g_*$.

Now, we want an analog of contractible spaces for our pairs: i.e. pairs of spaces (X, A) such that $H_n(X, A) = 0$. The correct definition is:

Definition 73.3.6. Let $A \subseteq X$. We say that A is a **deformation retract**[1] of X if there is a map of pairs $r \colon (X, A) \to (A, A)$ which is a pair-homotopy equivalence.

Example 73.3.7 (Examples of deformation retracts)

(a) If a single point p is a deformation retract of a space X, then X is contractible, since the retraction $r \colon X \to \{*\}$ (when viewed as a map $X \to X$) is homotopic to the identity map $\mathrm{id}_X \colon X \to X$.

(b) The punctured disk $D^2 \setminus \{0\}$ deformation retracts onto its boundary S^1.

(c) More generally, $D^n \setminus \{0\}$ deformation retracts onto its boundary S^{n-1}.

(d) Similarly, $\mathbb{R}^n \setminus \{0\}$ deformation retracts onto a sphere S^{n-1}.

Of course in this situation we have that

$$H_n(X, A) \cong H_n(A, A) = 0.$$

Exercise 73.3.8. Show that if $A \subseteq V \subseteq X$, and A is a deformation retract of V, then $H_n(X, A) \cong H_n(X, V)$ for all n. (Use Problem 72F*. Solution in next section.)

[1] This might be called a *deformation retraction in the weak sense* in other resources, such as [Ha02]

§73.4 Excision

Now for the key geometric result, which is the analog of Theorem 72.3.1 for our relative homology groups.

Theorem 73.4.1 (Excision)

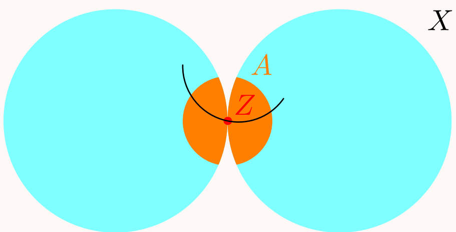
Let $Z \subseteq A \subseteq X$ be subspaces such that the closure of Z is contained in the interior of A. Then the inclusion $\iota(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ (viewed as a map of pairs) induces an isomorphism of relative homology groups

$$H_n(X \setminus Z, A \setminus Z) \cong H_n(X, A).$$

This means we can *excise* (delete) a subset Z of A in computing the relative homology groups $H_n(X, A)$. This should intuitively make sense: since we are "modding out by points in A", the internals of the set A should not matter so much.

Example 73.4.2

Excision may seem trivial (for a "relative cycle modulo relative boundary" in $H_n(X, A)$, just tweak the part that lies inside A until it doesn't touch Z), until you realize that it isn't always possible — you may accidentally cut a cycle apart! For example:



The main application of excision is to decide when $H_n(X, A) \cong \widetilde{H}_n(X/A)$. Answer:

Theorem 73.4.3 (Relative homology $\implies$ quotient space)

Let X be a space and A be a closed subspace such that A is a deformation retract of some open set $V \subseteq X$. Then the quotient map $q \colon X \to X/A$ induces an isomorphism

$$H_n(X, A) \cong H_n(X/A, A/A) \cong \widetilde{H}_n(X/A).$$

The key idea of the proof is: While it is not necessarily true that $H(X, A) \cong H(X/A, A/A)$ (indeed, we have seen two counterexamples earlier), if we cut out A, then we trivially have $H(X - A, A - A) \cong H(X/A - A/A, A/A - A/A)$. Unfortunately, this group is not isomorphic to $H(X, A)$, so we fix that using the set V — that is, $H(X - A, V - A) \cong H(X/A - A/A, V/A - A/A)$. The rest of the work is to use excision theorem and deformation retract to show the left hand side is isomorphic to $H(X, A)$, and the right hand side is isomorphic to $H(X/A)$.

Proof. By hypothesis, we can consider the following maps of pairs:

$$\begin{aligned} r &\colon (V, A) \to (A, A) \\ q &\colon (X, A) \to (X/A, A/A) \\ \widehat{q} &: (X - A, V - A) \to (X/A - A/A, V/A - A/A). \end{aligned}$$

Moreover, r is a pair-homotopy equivalence. Considering the long exact sequence of a triple (which was Problem 72F*) we have a diagram

$$\begin{tikzcd} H_n(V,A) \arrow[r] \arrow[d, "\cong"', "r"] & H_n(X,A) \arrow[r, "f"] & H_n(X,V) \arrow[r] & H_{n-1}(V,A) \arrow[d, "\cong"', "r"] \\ \underbrace{H_n(A,A)}_{=0} & & & \underbrace{H_{n-1}(A,A)}_{=0} \end{tikzcd}$$

where the isomorphisms arise since r is a pair-homotopy equivalence. So f is an isomorphism. Similarly the map

$$g \colon H_n(X/A, A/A) \to H_n(X/A, V/A)$$

is an isomorphism.

Now, consider the commutative diagram

$$\begin{tikzcd} H_n(X,A) \arrow[r, "f"] \arrow[d, "q_*"'] & H_n(X,V) & H_n(X - A, V - A) \arrow[l, "\text{Excise}"'] \arrow[d, "\cong"', "\widehat{q}_*"] \\ H_n(X/A, A/A) \arrow[r, "g"'] & H_n(X/A, V/A) & H_n(X/A - A/A, V/A - A/A) \arrow[l, "\text{Excise}"'] \end{tikzcd}$$

and observe that the rightmost arrow $\widehat{q}_*$ is an isomorphism, because outside of A the map $\widehat{q}$ is the identity. We know f and g are isomorphisms, as are the two arrows marked with "Excise" (by excision). From this we conclude that q_* is an isomorphism. Of course we already know that homology relative to a point is just the relative homology groups (this is the important case of Lemma 73.2.1). $\square$

§73.5 Some applications

One nice application of excision is to compute $\widetilde{H}_n(X \vee Y)$.

Theorem 73.5.1 (Homology of wedge sums)

Let X and Y be spaces with basepoints $x_0 \in X$ and $y_0 \in Y$, and assuming each point is a deformation retract of some open neighborhood. Then for every n we have

$$\widetilde{H}_n(X \vee Y) = \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y).$$

Proof. Apply Theorem 73.4.3 with the subset $\{x_0, y_0\}$ of $X \amalg Y$,

$$\begin{aligned} \widetilde{H}_n(X \vee Y) \cong \widetilde{H}_n((X \amalg Y)/\{x_0, y_0\}) &\cong H_n(X \amalg Y, \{x_0, y_0\}) \\ &\cong H_n(X, \{x_0\}) \oplus H_n(Y, \{y_0\}) \\ &\cong \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y). \end{aligned}$$

$\square$

Another application is to give a second method of computing $H_n(S^m)$. To do this, we will prove that

$$\widetilde{H}_n(S^m) \cong \widetilde{H}_{n-1}(S^{m-1})$$

for any $n, m > 1$. However,

- $\widetilde{H}_0(S^n)$ is $\mathbb{Z}$ for $n = 0$ and 0 otherwise.
- $\widetilde{H}_n(S^0)$ is $\mathbb{Z}$ for $m = 0$ and 0 otherwise.

So by induction on $\min\{m, n\}$ we directly obtain that

$$\widetilde{H}_n(S^m) \cong \begin{cases} \mathbb{Z} & m = n \\ 0 & \text{otherwise} \end{cases}$$

which is what we wanted.

To prove the claim, let's consider the exact sequence formed by the pair $X = D^2$ and $A = S^1$.

Example 73.5.2 (The long exact sequence for $(X, A) = (D^2, S^1)$)

Consider D^2 (which is contractible) with boundary S^1. Clearly S^1 is a deformation retraction of $D^2 \setminus \{0\}$, and if we fuse all points on the boundary together we get $D^2/S^1 \cong S^2$. So we have a long exact sequence

$$\begin{tikzcd}
\widetilde{H}_2(S^1) \arrow[r] & \underbrace{\widetilde{H}_2(D^2)}_{=0} \arrow[r] & \widetilde{H}_2(S^2) \arrow[dll] \\
\widetilde{H}_1(S^1) \arrow[r] & \underbrace{\widetilde{H}_1(D^2)}_{=0} \arrow[r] & \widetilde{H}_1(S^2) \arrow[dll] \\
\widetilde{H}_0(S^1) \arrow[r] & \underbrace{\widetilde{H}_0(D^2)}_{=0} \arrow[r] & \underbrace{\widetilde{H}_0(S^2)}_{=0}
\end{tikzcd}$$

From this diagram we read that

$$\dots, \quad \widetilde{H}_3(S^2) = \widetilde{H}_2(S^1), \quad \widetilde{H}_2(S^2) = \widetilde{H}_1(S^1), \quad \widetilde{H}_1(S^2) = \widetilde{H}_0(S^1).$$

More generally, the exact sequence for the pair $(X, A) = (D^m, S^{m-1})$ shows that $\widetilde{H}_n(S^m) \cong \widetilde{H}_{n-1}(S^{m-1})$, which is the desired conclusion.

§73.6 Invariance of dimension

Here is one last example of an application of excision.

Definition 73.6.1. Let X be a space and $p \in X$ a point. The kth **local homology group** of p at X is defined as

$$H_k(X, X \setminus \{p\}).$$

Note that for any open neighborhood U of p, we have by excision that

$$H_k(X, X \setminus \{p\}) \cong H_k(U, U \setminus \{p\}).$$

Thus this local homology group only depends on the space near p.

Theorem 73.6.2 (Invariance of dimension, Brouwer 1910)

Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be nonempty open sets. If U and V are homeomorphic, then $m = n$.

Proof. Consider a point $x \in U$ and its local homology groups. By excision,

$$H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong H_k(U, U \setminus \{x\}).$$

But since $\mathbb{R}^n \setminus \{x\}$ is homotopic to S^{n-1}, the long exact sequence of Theorem 72.2.5 tells us that

$$H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & \text{otherwise.} \end{cases}$$

Analogously, given $y \in V$ we have

$$H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{y\}) \cong H_k(V, V \setminus \{y\}).$$

If $U \cong V$, we thus deduce that

$$H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{y\})$$

for all k. This of course can only happen if $m = n$. $\square$

§73.7 A few harder problems to think about

Problem 73A. Let $X = S^1 \times S^1$ and $Y = S^1 \vee S^1 \vee S^2$. Show that

$$H_n(X) \cong H_n(Y)$$

for every integer n.

Problem 73B (Hatcher §2.1 exercise 18)**.** Consider $\mathbb{Q} \subset \mathbb{R}$. Compute $\widetilde{H}_1(\mathbb{R}, \mathbb{Q})$.

Problem 73C*. What are the local homology groups of a topological n-manifold?

Problem 73D. Let

$$X = \{(x, y) \mid x \ge 0\} \subseteq \mathbb{R}^2$$

denote the half-plane. What are the local homology groups of points in X?

Problem 73E (Brouwer-Jordan separation theorem, generalizing Jordan curve theorem)**.** Let $X \subseteq \mathbb{R}^n$ be a subset which is homeomorphic to S^{n-1}. Prove that $\mathbb{R}^n \setminus X$ has exactly two path-connected components.

74 Bonus: Cellular homology

We now introduce cellular homology, which essentially lets us compute the homology groups of any CW complex we like.

§74.1 Degrees

Prototypical example for this section: $z \mapsto z^d$ has degree d.

For any $n > 0$ and map $f\colon S^n \to S^n$, consider

$$f_*\colon \underbrace{H_n(S^n)}_{\cong\mathbb{Z}} \to \underbrace{H_n(S^n)}_{\cong\mathbb{Z}}$$

which must be multiplication by some constant d. This d is called the **degree** of f, denoted $\deg f$.

Question 74.1.1. Show that $\deg(f \circ g) = \deg(f)\deg(g)$.

As we mentioned in Example 71.2.12, roughly speaking:

$\deg f$ counts how many times $\operatorname{im} f$ wraps around S^n.

Or, it counts how many "S^n bags" that $\operatorname{im} f$ consists of.

Example 74.1.2 (Degree)

(a) For $n = 1$, the map $z \mapsto z^k$ (viewing $S^1 \subseteq \mathbb{C}$) has degree k.

(b) A reflection map $(x_0, x_1, \dots, x_n) \mapsto (-x_0, x_1, \dots, x_n)$ has degree -1; we won't prove this, but geometrically this should be clear.

(c) The antipodal map $x \mapsto -x$ has degree $(-1)^{n+1}$ since it's the composition of $n+1$ reflections as above. We denote this map by $-\mathrm{id}$.

Obviously, if f and g are homotopic, then $\deg f = \deg g$. In fact, a theorem of Hopf says that this is a classifying invariant: anytime $\deg f = \deg g$, we have that f and g are homotopic.

One nice application of this:

Theorem 74.1.3 (Hairy ball theorem)

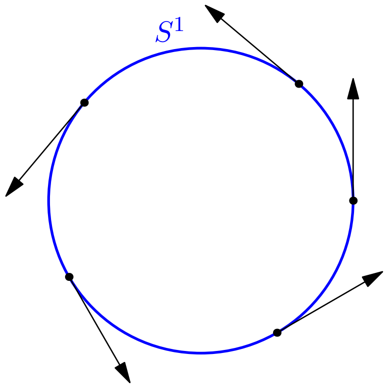
If $n > 0$ is even, then S^n doesn't have a continuous field of nonzero tangent vectors.

Proof. If the vectors are nonzero then WLOG they have norm 1; that is for every x we have an orthogonal unit vector $v(x)$. Then we can construct a homotopy map $F\colon S^n \times [0,1] \to S^n$ by

$$(x,t) \mapsto (\cos \pi t)x + (\sin \pi t)v(x).$$

which gives a homotopy from id to $-\text{id}$. So $\deg(\text{id}) = \deg(-\text{id})$, which means $1 = (-1)^{n+1}$ so n must be odd. $\square$

Of course, the one can construct such a vector field whenever n is odd. For example, when $n = 1$ such a vector field is drawn below.



§74.2 Cellular chain complex

Before starting, we state:

Lemma 74.2.1 (CW homology groups)
Let X be a CW complex. Then

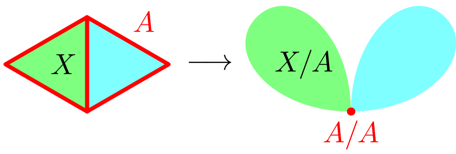
$$H_k(X^n, X^{n-1}) \cong \begin{cases} \mathbb{Z}^{\oplus \#n\text{-cells of } X} & k = n \\ 0 & \text{otherwise.} \end{cases}$$

and

$$H_k(X^n) \cong \begin{cases} H_k(X) & k \le n-1 \\ 0 & k \ge n+1. \end{cases}$$

Proof. The first part is immediate by noting that (X^n, X^{n-1}) satisfies the hypothesis of Theorem 73.4.3, so $H_k(X^n, X^{n-1}) \cong \widetilde{H}_k(X^n/X^{n-1})$, and X^n/X^{n-1} is a wedge sum of several n-spheres.

For an example, for $n = 2$ (the "spheres" are drawn as a balloon-shaped blob here):



For the second part, fix k and note that, as long as $n \le k-1$ or $n \ge k+2$,

$$\underbrace{H_{k+1}(X^n, X^{n-1})}_{=0} \to H_k(X^{n-1}) \to H_k(X^n) \to \underbrace{H_k(X^n, X^{n-1})}_{=0}.$$

So we have isomorphisms

$$H_k(X^{k-1}) \cong H_k(X^{k-2}) \cong \ldots \cong H_k(X^0) = 0$$

and

$$H_k(X^{k+1}) \cong H_k(X^{k+2}) \cong \dots \cong H_k(X). \qquad \square$$

So, we know that the groups $H_k(X^k, X^{k-1})$ are super nice: they are free abelian with basis given by the cells of X. So, we give them a name:

Definition 74.2.2. For a CW complex X, we define

$$\operatorname{Cells}_k(X) = H_k(X^k, X^{k-1})$$

where $\operatorname{Cells}_0(X) = H_0(X^0, \varnothing) = H_0(X^0)$ by convention. So $\operatorname{Cells}_k(X)$ is an abelian group with basis given by the k-cells of X.

Now, using $\operatorname{Cells}_k = H_k(X^k, X^{k-1})$ let's use our long exact sequence and try to string together maps between these. Consider the following diagram.

$$\begin{tikzcd}
 & \underbrace{H_3(X^2)}_{=0} \arrow[d, "0"] & & & & \\
\boxed{\operatorname{Cells}_4(X)} \arrow[r, "\partial_4"] \arrow[dr, "d_4"'] & H_3(X^3) \arrow[r, two heads] \arrow[d, hook] & \underbrace{H_3(X^4)}_{\cong H_3(X)} \arrow[r, "0"] & \underbrace{H_3(X^4, X^3)}_{=0} & & \\
 & \boxed{\operatorname{Cells}_3(X)} \arrow[d, "\partial_3"] \arrow[dr, "d_3"] & & \underbrace{H_1(X^0)}_{=0} \arrow[d, "0"] & & \\
\underbrace{H_2(X^1)}_{=0} \arrow[r, "0"] & H_2(X^2) \arrow[r, hook] \arrow[d, two heads] & \boxed{\operatorname{Cells}_2(X)} \arrow[r, "\partial_2"] \arrow[dr, "d_2"] & H_1(X^1) \arrow[r, two heads] \arrow[d, hook] & \underbrace{H_1(X^2)}_{\cong H_1(X)} \arrow[r, "0"] & \underbrace{H_1(X^2, X^1)}_{=0} \\
 & \underbrace{H_2(X^3)}_{\cong H_2(X)} \arrow[d, "0"] & & \boxed{\operatorname{Cells}_1(X)} \arrow[d, "\partial_1"] \arrow[dr, "d_1"] & & \\
 & \underbrace{H_2(X^3, X^2)}_{=0} & \underbrace{H_0(\varnothing)}_{=0} \arrow[r, "0"] & H_0(X^0) \arrow[r, hook] \arrow[d, two heads] & \boxed{\operatorname{Cells}_0(X)} \arrow[r, "\partial_0"] & \dots \\
 & & & \underbrace{H_0(X^1)}_{\cong H_0(X)} \arrow[d, "0"] & & \\
 & & & \underbrace{H_0(X^1, X^0)}_{=0} & &
\end{tikzcd}$$

The idea is that we have taken all the exact sequences generated by adjacent skeletons, and strung them together at the groups $H_k(X^k)$, with half the exact sequences being laid out vertically and the other half horizontally.

In that case, composition generates a sequence of blue maps between the $H_k(X^k, X^{k-1})$ as shown.

Question 74.2.3. Show that the composition of two adjacent blue arrows is zero.

So from the diagram above, we can read off a sequence of arrows

$$\dots \xrightarrow{d_5} \operatorname{Cells}_4(X) \xrightarrow{d_4} \operatorname{Cells}_3(X) \xrightarrow{d_3} \operatorname{Cells}_2(X) \xrightarrow{d_2} \operatorname{Cells}_1(X) \xrightarrow{d_1} \operatorname{Cells}_0(X) \xrightarrow{d_0} 0.$$

This is a chain complex, called the **cellular chain complex**; as mentioned before all the homology groups are free, but these ones are especially nice because for most reasonable CW complexes, they are also finitely generated (unlike the massive $C_\bullet(X)$ that we had earlier). In other words, the $H_k(X^k, X^{k-1})$ are especially nice "concrete" free groups that one can actually work with.

The other reason we care is that in fact:

Theorem 74.2.4 (Cellular chain complex gives $H_n(X)$)
The kth homology group of the cellular chain complex is isomorphic to $H_k(X)$.

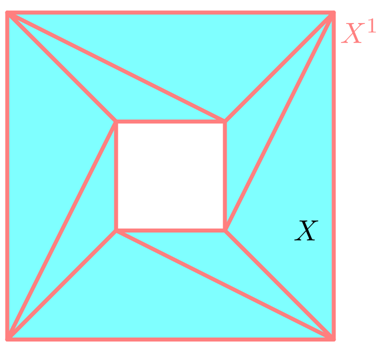
Proof. Follows from the diagram; Problem 74D. □

§74.3 Digression: why are the homology groups equal?

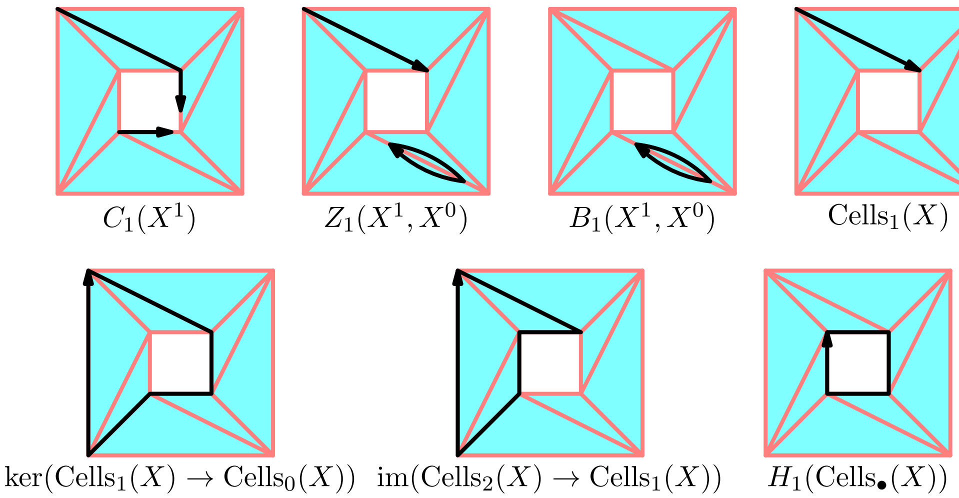
There is another intuition that explains it — roughly speaking,

$$H_k(\operatorname{Cells}_\bullet(X)) = \frac{\text{aligned cycle}}{\text{aligned boundary}} = \frac{\text{aligned cycle} \times \text{fuzz}}{\text{aligned boundary} \times \text{fuzz}} = \frac{\text{cycle}}{\text{boundary}} = H_k(X).$$

Let me explain. Consider a CW-complex X that looks like the following, where X^1 is drawn in red. Each blue region corresponds to a 2-cell.



Then, look at the following figure.



It looks like a lot, so let me explain.

- The first picture depicts a typical element of $C_1(X^1)$ — that is, a 1-chain that is contained in X^1, being the formal sum of two maps from Δ^1 to X^1, whose image is drawn as black arrows.

  Note that only the image of the maps are depicted, information such as which point of the simplex Δ^1 get mapped to which point inside X^1 is not shown — although different continuous maps give rise to different elements of $C_1(X^1)$.

- The second picture depicts a typical element of $Z_1(X^1, X^0)$ — that is, the *relative cycles*.

  Although we never formally defined what is a relative cycle or the groups $Z_1(X^1, X^0)$, you can guess the definition from the definition of $Z_1(X^1)$ — it is the subgroup of $C_1(X^1, X^0) = C_1(X^1)/C_1(X^0)$ whose boundary vanish.

  The fact that the loop on the bottom is flattened is just to make it look nicer — the whole thing is contained inside the red skeleton i.e. X^1.

  Of course, being an element of the quotient, only a representative element is depicted — the "modded out" parts are the chains that are entirely contained inside X^0 i.e. some vertices.

- The third picture depicts a typical element of $B_1(X^1, X^0)$ i.e. the *relative boundaries*.

  This belongs to $\mathrm{im}\left(C_2(X^1, X^0) \xrightarrow{\partial} C_1(X^1, X^0)\right)$ — in words, there is some 2-chain whose boundary equals the depicted element.

- The fourth picture depicts a typical element of $\mathrm{Cells}_1(X)$ — that is, "relative cycles mod relative boundaries".

  Hopefully it is intuitively obvious how this group is isomorphic to the abelian group generated by each 1-cell of X.

  We can in fact think of each of these elements as an "aligned element" of $C_1(X^1)$ where all endpoints lie inside a vertex (that is, the boundary of that element is inside X^0), and for each 1-cell, a canonical 1-simplex is chosen to cover that cell (note that different simplexes with the same image intuitively corresponds to "reparametrization" to change the speed, and the difference between a simplex and its reparametrization is in fact an element of $B_1(X^1, X^0)$ — try to make this rigorous! Hint: use the prism operator.)

- The fifth picture depicts a typical element of

$$\ker\left(\text{Cells}_1(X) \xrightarrow{\partial} \text{Cells}_0(X)\right)$$

which can be thought of as a "cellular cycle", or a 1-cycle (element of $Z_1(X)$) that is "aligned", as explained above.

- The sixth picture depicts a typical element of

$$\operatorname{im}\left(\text{Cells}_2(X) \xrightarrow{\partial} \text{Cells}_1(X)\right)$$

which can be thought of as a "cellular boundary", or a 1-boundary (element of $B_1(X)$[1]) that is "aligned" in the same sense as above.

- Finally, the last picture is $H_1(\text{Cells}_\bullet(X))$, which is

$$H_1(\text{Cells}_\bullet(X)) = \frac{\ker\left(\text{Cells}_1(X) \xrightarrow{\partial} \text{Cells}_0(X)\right)}{\operatorname{im}\left(\text{Cells}_2(X) \xrightarrow{\partial} \text{Cells}_1(X)\right)}.$$

Or, roughly speaking,

$$H_1(\text{Cells}_\bullet(X)) = \frac{\text{aligned cycle}}{\text{aligned boundary}}.$$

That is what we mean by $\frac{\text{aligned cycle}}{\text{aligned boundary}} = \frac{\text{cycle}}{\text{boundary}}$. With a suitable formalization and arbitrary selection of canonical simplices,[2] we can make the argument above rigorous.

What do we mean by "fuzz"? This part is hopefully obvious, but the point is that an aligned cycle can be "moved around" a bit (with reparametrization, or addition of elements in $B_1(X^1, X^0)$) while still keep it a cycle (that is, an element of $Z_1(X)$). Similarly for aligned boundaries.

So, the point is — we can "cancel" the common fuzz factor in the numerator and the denominator, and the result will remain the same.

Refer to [**Ha02**] for some formal treatment on simplicial approximation.

§74.4 Application: Euler characteristic via Betti numbers

A nice application of this is to define the **Euler characteristic** of a finite CW complex X. Of course we can write

$$\chi(X) = \sum_n (-1)^n \cdot \#(n\text{-cells of } X)$$

which generalizes the familiar $V-E+F$ formula. However, this definition is unsatisfactory because it depends on the choice of CW complex, while we actually want $\chi(X)$ to only depend on the space X itself (and not how it was built). In light of this, we prove that:

Theorem 74.4.1 (Euler characteristic via Betti numbers)
For any finite CW complex X we have

$$\chi(X) = \sum_n (-1)^n \operatorname{rank} H_n(X).$$

[1] This is not an element of $B_1(X^1)$! Think about why.

[2] Technically we need a so-called Δ-complex structure on X, but we don't define Δ-structure in the Napkin. See [**Ha02**] for details.

Thus $\chi(X)$ does not depend on the choice of CW decomposition. The numbers

$$b_n = \operatorname{rank} H_n(X)$$

are called the **Betti numbers** of X. In fact, we can use this to define $\chi(X)$ for any reasonable space; we are happy because in the (frequent) case that X is a CW complex, the definition coincides with the normal definition of the Euler characteristic.

Proof. We quote the fact that if $0 \to A \to B \to C \to D \to 0$ is exact then $\operatorname{rank} B + \operatorname{rank} D = \operatorname{rank} A + \operatorname{rank} C$. Then for example the row

$$\underbrace{H_2(X^1)}_{=0} \xrightarrow{0} H_2(X^2) \hookrightarrow H_2(X^2, X^1) \xrightarrow{\partial_2} H_1(X^1) \twoheadrightarrow \underbrace{H_1(X^2)}_{\cong H_1(X)} \xrightarrow{0} \underbrace{H_1(X^2, X^1)}_{=0}$$

from the cellular diagram gives

$$\#(\text{2-cells}) + \operatorname{rank} H_1(X) = \operatorname{rank} H_2(X^2) + \operatorname{rank} H_1(X^1).$$

More generally,

$$\#(k\text{-cells}) + \operatorname{rank} H_{k-1}(X) = \operatorname{rank} H_k(X^k) + \operatorname{rank} H_{k-1}(X^{k-1})$$

which holds also for $k = 0$ if we drop the H_{-1} terms (since $\#\text{0-cells} = \operatorname{rank} H_0(X^0)$ is obvious). Multiplying this by $(-1)^k$ and summing across $k \geq 0$ gives the conclusion. $\square$

Example 74.4.2 (Examples of Betti numbers)

(a) The Betti numbers of S^n are $b_0 = b_n = 1$, and zero elsewhere. The Euler characteristic is $1 + (-1)^n$.

(b) The Betti numbers of a torus $S^1 \times S^1$ are $b_0 = 1$, $b_1 = 2$, $b_2 = 1$, and zero elsewhere. Thus the Euler characteristic is 0.

(c) The Betti numbers of $\mathbb{CP}^n$ are $b_0 = b_2 = \cdots = b_{2n} = 1$, and zero elsewhere. Thus the Euler characteristic is $n + 1$.

(d) The Betti numbers of the Klein bottle are $b_0 = 1$, $b_1 = 1$ and zero elsewhere. Thus the Euler characteristic is 0, the same as the sphere (also since their CW structures use the same number of cells).

One notices that in the "nice" spaces S^n, $S^1 \times S^1$ and $\mathbb{CP}^n$ there is a nice symmetry in the Betti numbers, namely $b_k = b_{n-k}$. This is true more generally; see Poincaré duality and Problem 76A†.

§74.5 The cellular boundary formula

In fact, one can describe explicitly what the maps d_n are. Recalling that $H_k(X^k, X^{k-1})$ has a basis the k-cells of X, we obtain:

Theorem 74.5.1 (Cellular boundary formula for $k = 1$)

For $k = 1$,

$$d_1 \colon \operatorname{Cells}_1(X) \to \operatorname{Cells}_0(X)$$

is just the boundary map.

Theorem 74.5.2 (Cellular boundary for $k > 1$)
Let $k > 1$ be a positive integer. Let e^k be an k-cell, and let $\{e_\beta^{k-1}\}_\beta$ denote all $(k-1)$-cells of X. Then

$$d_k \colon \operatorname{Cells}_k(X) \to \operatorname{Cells}_{k-1}(X)$$

is given on basis elements by

$$d_k(e^k) = \sum_\beta d_\beta e_\beta^{k-1}$$

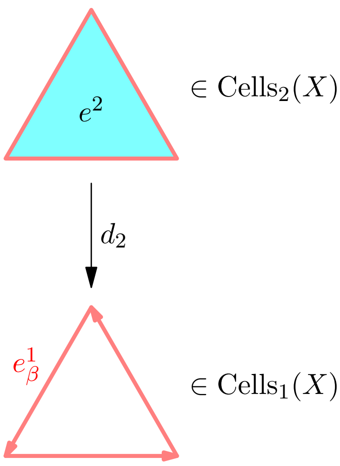
where d_β is be the degree of the composed map

$$S^{k-1} = \partial e^k \xrightarrow{\text{attach}} X^{k-1} \twoheadrightarrow S_\beta^{k-1}.$$

Here the first arrow is the attaching map for e^k and the second arrow is the quotient of collapsing $X^{k-1} \setminus e_\beta^{k-1}$ to a point.

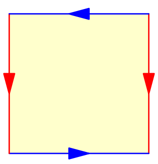
What is the degree doing here? Remember that a basis element $e^k \in \operatorname{Cells}_k(X)$ is just a k-cell, and its boundary should be just the cells that forms its boundary.

With the same visualization as above, we can do something like the following.



But it's not that easy! Note that in a CW complex, the boundary of a k-cell can be fused into *arbitrary points* in X^{k-1}, so an "edge" of a k-cell need not be a $k-1$-cell.

To make matters worse, sometimes there may be a duplicated edge — in the Klein bottle, each pair of two opposing edges depicted actually *the same edge*, possibly in different orientations.



In such a case, we need to count the *multiplicity* of each edge — and this is exactly what the degree of the map counts! We will see an explicit example of computing the homology groups of the Klein bottle in just a moment.

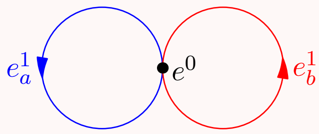
This gives us an algorithm for computing homology groups of a CW complex:

- Construct the cellular chain complex, where $\text{Cells}_k(X)$ is $\mathbb{Z}^{\oplus \# k\text{-cells}}$.
- $d_1 \colon \text{Cells}_1(X) \to \text{Cells}_0(X)$ is just the boundary map (so $d_1(e^1)$ is the difference of the two endpoints).
- For any $k > 1$, we compute $d_k \colon \text{Cells}_k(X) \to \text{Cells}_{k-1}(X)$ on basis elements as follows. Repeat the following for each k-cell e^k:
  - For every $k-1$ cell e^{k-1}_β, compute the degree of the boundary of e^k welded onto the boundary of e^{k-1}_β, say d_β.
  - Then $d_k(e^k) = \sum_\beta d_\beta e^{k-1}_\beta$.
- Now we have the maps of the cellular chain complex, so we can compute the homologies directly (by taking the quotient of the kernel by the image).

We can use this for example to compute the homology groups of the torus again, as well as the Klein bottle and other spaces.

Example 74.5.3 (Cellular homology of a torus)

Consider the torus built from e^0, e^1_a, e^1_b and e^2 as before, where e^2 is attached via the word $aba^{-1}b^{-1}$. For example, X^1 is



The cellular chain complex is

$$0 \longrightarrow \mathbb{Z}e^2 \xrightarrow{d_2} \mathbb{Z}e^1_a \oplus \mathbb{Z}e^1_b \xrightarrow{d_1} \mathbb{Z}e^0 \xrightarrow{d_0} 0$$

Now apply the cellular boundary formulas:

- Recall that d_1 was the boundary formula. We have $d_1(e^1_a) = e_0 - e_0 = 0$ and similarly $d_1(e^1_b) = 0$. So $d_1 = 0$.
- For d_2, consider the image of the boundary e^2 on e^1_a. Around X^1, it wraps once around e^1_a, once around e^1_b, again around e^1_a (in the opposite direction), and again around e^1_b. Once we collapse the entire e^1_b to a point, we see that the degree of the map is 0. So $d_2(e^2)$ has no e^1_a coefficient. Similarly, it has no e^1_b coefficient, hence $d_2 = 0$.

Thus

$$d_1 = d_2 = 0.$$

So at every map in the complex, the kernel of the map is the whole space while the image is $\{0\}$. So the homology groups are $\mathbb{Z}$, $\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}$.

Example 74.5.4 (Cellular homology of the Klein bottle)

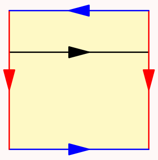
Let X be a Klein bottle. Consider cells e^0, e^1_a, e^1_b and e^2 as before, but this time e^2 is attached via the word $abab^{-1}$. So d_1 is still zero, but this time we have $d_2(e^2) = 2e^1_a$ instead (why?). So our diagram looks like

$$\begin{array}{ccccccccc} 0 & \xrightarrow{0} & \mathbb{Z}e^2 & \xrightarrow{d_2} & \mathbb{Z}e_a^1 \oplus \mathbb{Z}e_b^1 & \xrightarrow{d_1} & \mathbb{Z}e^0 & \xrightarrow{d_0} & 0 \\ & & e^2 & \longmapsto & 2e_a^1 & & & & \\ & & & & e_1^a & \longmapsto & 0 & & \\ & & & & e_1^b & \longmapsto & 0 & & \end{array}$$

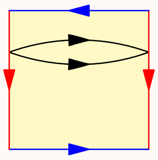
So we get that $H_0(X) \cong \mathbb{Z}$, but

$$H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

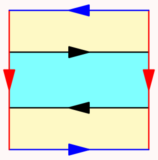
this time (it is $\mathbb{Z}^{\oplus 2}$ modulo a copy of $2\mathbb{Z}$). Also, $\ker d_2 = 0$, and so now $H_2(X) = 0$. Let us sanity check that this makes sense — that is, there is some cycle that is not a boundary, but when doubled it become a boundary.
Actually, most cycles work.



If we double up the path, we get something like the following.



Here is the important part: since the two blue edges are identified in opposite direction, we can pull one of the path across the edge to reverse its direction. . . but now the region is in fact the boundary of the cyan region! So we're done.



It remains to convince yourself that the difference of two homotopy equivalent path is a boundary.

§74.6 A few harder problems to think about

Problem 74A†. Let n be a positive integer. Show that

$$H_k(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & k = 0, 2, 4, \dots, 2n \\ 0 & \text{otherwise.} \end{cases}$$

Problem 74B. Show that a non-surjective map $f \colon S^n \to S^n$ has degree zero.

Problem 74C (Moore spaces)**.** Let $G_1, G_2, \dots, G_N$ be a sequence of finitely generated abelian groups. Construct a space X such that

$$\widetilde{H}_n(X) \cong \begin{cases} G_n & 1 \le n \le N \\ 0 & \text{otherwise.} \end{cases}$$

Problem 74D. Prove Theorem 74.2.4, showing that the homology groups of X coincide with the homology groups of the cellular chain complex.

Problem 74E†. Let n be a positive integer. Show that

$$H_k(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } k = n \equiv 1 \pmod 2 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } k \text{ is odd and } 0 < k < n \\ 0 & \text{otherwise.} \end{cases}$$

75 Singular cohomology

Here's one way to motivate this chapter. It turns out that:

- $H_n(\mathbb{CP}^2) \cong H_n(S^2 \vee S^4)$ for every n.
- $H_n(\mathbb{CP}^3) \cong H_n(S^2 \times S^4)$ for every n.

This is unfortunate, because if possible we would like to be able to tell these spaces apart (as they are in fact not homotopy equivalent), but the homology groups cannot tell the difference between them.

In this chapter, we'll define a *cohomology group* $H^n(X)$ and $H^n(Y)$. In fact, the H^n's are completely determined by the H_n's by the so-called *universal coefficient theorem*. However, it turns out that one can take all the cohomology groups and put them together to form a *cohomology ring* $H^\bullet$.[1] We will then see that $H^\bullet(X) \not\cong H^\bullet(Y)$ as rings.

§75.1 Cochain complexes

Definition 75.1.1. A **cochain complex** $A^\bullet$ is algebraically the same as a chain complex, except that the indices increase. So it is a sequence of abelian groups

$$\dots \xrightarrow{\delta} A^{n-1} \xrightarrow{\delta} A^n \xrightarrow{\delta} A^{n+1} \xrightarrow{\delta} \dots.$$

such that $\delta^2 = 0$. Notation-wise, we're now using superscripts, and use δ rather ∂. We define the **cohomology groups** by

$$H^n(A^\bullet) = \ker\left(A^n \xrightarrow{\delta} A^{n+1}\right) / \operatorname{im}\left(A^{n-1} \xrightarrow{\delta} A^n\right).$$

Example 75.1.2 (de Rham cohomology)

We have already met one example of a cochain complex: let M be a smooth manifold and $\Omega^k(M)$ be the additive group of k-forms on M. Then we have a cochain complex

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \dots.$$

The resulting cohomology is called **de Rham cohomology**, described later.

Aside from de Rham's cochain complex, **the most common way to get a cochain complex is to *dualize* a chain complex.** Specifically, pick an abelian group G; note that $\operatorname{Hom}(-, G)$ is a contravariant functor, and thus takes every chain complex

$$\dots \xrightarrow{\partial} A_{n+1} \xrightarrow{\partial} A_n \xrightarrow{\partial} A_{n-1} \xrightarrow{\partial} \dots$$

into a cochain complex: letting $A^n = \operatorname{Hom}(A_n, G)$ we obtain

$$\dots \xrightarrow{\delta} A^{n-1} \xrightarrow{\delta} A^n \xrightarrow{\delta} A^{n+1} \xrightarrow{\delta} \dots.$$

[1] [Ha02] has an explanation why is it that cohomology has more structures than homology — roughly speaking, the natural maps $X \times X \to X$ must be a projection which is not very interesting, but there is a more interesting natural map $X \to X \times X$ given by $p \mapsto (p, p)$.

where $\delta(A_n \xrightarrow{f} G) = A_{n+1} \xrightarrow{\partial} A \xrightarrow{f} G$.

These are the cohomology groups we study most in algebraic topology, so we give a special notation to them.

Definition 75.1.3. Given a chain complex $A_\bullet$ of abelian groups and another group G, we let

$$H^n(A_\bullet; G)$$

denote the cohomology groups of the dual cochain complex $A^\bullet$ obtained by applying $\operatorname{Hom}(-, G)$. In other words, $H^n(A_\bullet; G) = H^n(A^\bullet)$.

§75.2 Cohomology of spaces

Prototypical example for this section: $C^0(X; G)$ all functions $X \to G$ while $H^0(X)$ are those functions $X \to G$ constant on path components.

The case of interest is our usual geometric situation, with $C_\bullet(X)$.

Definition 75.2.1. For a space X and abelian group G, we define $C^\bullet(X; G)$ to be the dual to the singular chain complex $C_\bullet(X)$, called the **singular cochain complex** of X; its elements are called **cochains**.

Then we define the **cohomology groups** of the space X as

$$H^n(X; G) := H^n(C_\bullet(X); G) = H^n(C^\bullet(X; G)).$$

Remark 75.2.2 — Note that if G is also a ring (like $\mathbb{Z}$ or $\mathbb{R}$), then $H^n(X; G)$ is not only an abelian group but actually a G-module.

Example 75.2.3 ($C^0(X; G)$, $C^1(X; G)$, and $H^0(X; G)$)

Let X be a topological space and consider $C^\bullet(X)$.

- $C_0(X)$ is the free abelian group on X, and $C^0(X) = \operatorname{Hom}(C_0(X), G)$. So a 0-cochain is a function that takes every point of X to an element of G.
- $C_1(X)$ is the free abelian group on 1-simplices in X. So $C^1(X)$ needs to take every 1-simplex to an element of G.

Let's now try to understand $\delta\colon C^0(X) \to C^1(X)$. Given a 0-cochain $\phi \in C^0(X)$, i.e. a homomorphism $\phi\colon C^0(X) \to G$, what is $\delta\phi\colon C^1(X) \to G$? Answer:

$$\delta\phi\colon [v_0, v_1] \mapsto \phi([v_0]) - \phi([v_1]).$$

Hence, elements of $\ker(C^0 \xrightarrow{\delta} C^1) \cong H^0(X; G)$ are those cochains that are *constant on path-connected components.*

In particular, much like $H_0(X)$, we have

$$H^0(X) \cong G^{\oplus r}$$

if X has r path-connected components (where r is finite[2]).

Abuse of Notation 75.2.4. In this chapter the only cochain complexes we will consider are dual complexes as above. So, any time we write a cochain complex $A^\bullet$ it is implicitly given by applying $\operatorname{Hom}(-, G)$ to $A_\bullet$.

The higher cohomology groups $H^n(X; G)$ (or even the cochain groups $C^n(X; G) = \operatorname{Hom}(C_n(X), G)$) are harder to describe concretely.

§75.3 Cohomology of spaces is functorial

We now check that the cohomology groups still exhibit the same nice functorial behavior. First, let's categorize the previous results we had:

Question 75.3.1. Define $\mathsf{CoCmplx}$ the category of cochain complexes.

Exercise 75.3.2. Interpret $\operatorname{Hom}(-, G)$ as a contravariant functor from

$$\operatorname{Hom}(-, G) \colon \mathsf{Cmplx}^{\mathrm{op}} \to \mathsf{CoCmplx}.$$

This means in particular that given a chain map $f \colon A_\bullet \to B_\bullet$, we naturally obtain a dual map $f^\vee \colon B^\bullet \to A^\bullet$.

Question 75.3.3. Interpret $H^n \colon \mathsf{CoCmplx} \to \mathsf{Grp}$ as a functor. Compose these to get a contravariant functor $H^n(-; G) \colon \mathsf{Cmplx}^{\mathrm{op}} \to \mathsf{Grp}$.

Then in exact analog to our result that $H_n \colon \mathsf{hTop} \to \mathsf{Grp}$ we have:

Theorem 75.3.4 ($H^n(-; G) : \mathsf{hTop}^{\mathrm{op}} \to \mathsf{Grp}$)
For every n, $H^n(-; G)$ is a contravariant functor from $\mathsf{hTop}^{\mathrm{op}}$ to Grp.

Proof. The idea is to leverage the work we already did in constructing the prism operator earlier. First, we construct the entire sequence of functors from $\mathsf{Top}^{\mathrm{op}} \to \mathsf{Grp}$:

[2] Something funny happens if X has *infinitely* many path-connected components: say $X = \coprod_\alpha X_\alpha$ over an infinite indexing set. In this case we have $H_0(X) = \bigoplus_\alpha G$ while $H^0(X) = \prod_\alpha G$. For homology we get a *direct sum* while for cohomology we get a *direct product*.

These are actually different for infinite indexing sets. For general modules $\bigoplus_\alpha M_\alpha$ is *defined* to only allow to have *finitely many* nonzero terms. (This was never mentioned earlier in the Napkin, since I only ever defined $M \oplus N$ and extended it to finite direct sums.) No such restriction holds for $\prod_\alpha G_\alpha$ a product of groups. This corresponds to the fact that $C_0(X)$ is formal linear sums of 0-chains (which, like all formal sums, are finite) from the path-connected components of G. But a cochain of $C^0(X)$ is a *function* from each path-connected component of X to G, where there is no restriction.

$$\begin{tikzcd}
\mathsf{Top}^{\mathrm{op}} \arrow[r, "C_\bullet"] & \mathsf{Cmplx}^{\mathrm{op}} \arrow[r, "\mathrm{Hom}(-;G)"] & \mathsf{CoCmplx} \arrow[r, "H^n"] & \mathsf{Grp}
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[d, "f"'] & & C_\bullet(X) \arrow[d, "f_\sharp"'] & & C^\bullet(X;G) & & H^n(X;G) \\
Y \arrow[r, phantom, ""] & \arrow[r, mapsto] & C_\bullet(Y) & \arrow[r, mapsto] & C^\bullet(Y;G) \arrow[u, "f_\sharp"'] & \arrow[r, mapsto] & H^n(Y;G). \arrow[u, "f^*"']
\end{tikzcd}$$

Here $f^\sharp = (f_\sharp)^\vee$, and f^* is the resulting induced map on homology groups of the cochain complex.

So as before all we have to show is that $f \simeq g$, then $f^* = g^*$. Recall now that there is a prism operator such that $f_\sharp - g_\sharp = P\partial + \partial P$. If we apply the entire functor $\operatorname{Hom}(-;G)$ we get that $f^\sharp - g^\sharp = \delta P^\vee + P^\vee \delta$ where $P^\vee \colon C^{n+1}(Y;G) \to C^n(X;G)$. So $f^\sharp$ and $g^\sharp$ are chain homotopic thus $f^* = g^*$. $\square$

§75.4 Universal coefficient theorem

We now wish to show that the cohomology groups are determined up to isomorphism by the homology groups: given $H_n(A_\bullet)$, we can extract $H^n(A_\bullet;G)$. This is achieved by the *universal coefficient theorem*.

Theorem 75.4.1 (Universal coefficient theorem)

Let $A_\bullet$ be a chain complex of *free* abelian groups, and let G be another abelian group. Then there is a natural short exact sequence

$$0 \to \operatorname{Ext}(H_{n-1}(A_\bullet), G) \to H^n(A_\bullet;G) \xrightarrow{h} \operatorname{Hom}(H_n(A_\bullet), G) \to 0.$$

In addition, this exact sequence is *split* so in particular

$$H^n(C_\bullet;G) \cong \operatorname{Ext}(H_{n-1}(A_\bullet), G) \oplus \operatorname{Hom}(H_n(A_\bullet), G).$$

Fortunately, in our case of interest, $A_\bullet$ is $C_\bullet(X)$ which is by definition free.

There are two things we need to explain, what the map h is and the map Ext is.

It's not too hard to guess how

$$h \colon H^n(A_\bullet;G) \to \operatorname{Hom}(H_n(A_\bullet), G)$$

is defined. An element of $H^n(A_\bullet;G)$ is represented by a function which sends a cycle in A_n to an element of G. The content of the theorem is to show that h is surjective with kernel $\operatorname{Ext}(H_{n-1}(A_\bullet), G)$.

What about Ext? It turns out that $\operatorname{Ext}(-,G)$ is the so-called **Ext functor**, defined as follows. Let H be an abelian group, and consider a **free resolution** of H, by which we mean an exact sequence

$$\dots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

with each F_i free. Then we can apply $\operatorname{Hom}(-,G)$ to get a cochain complex

$$\dots \xleftarrow{f_2^\vee} \operatorname{Hom}(F_1,G) \xleftarrow{f_1^\vee} \operatorname{Hom}(F_0,G) \xleftarrow{f_0^\vee} \operatorname{Hom}(H,G) \leftarrow 0.$$

but *this cochain complex need not be exact* (in categorical terms, $\operatorname{Hom}(-, G)$ does not preserve exactness). We define

$$\operatorname{Ext}(H, G) := \ker(f_2^\vee)/\operatorname{im}(f_1^\vee)$$

and it's a theorem that this doesn't depend on the choice of the free resolution. There's a lot of homological algebra that goes into this, which I won't take the time to discuss; but the upshot of the little bit that I did include is that the Ext functor is very easy to compute in practice, since you can pick any free resolution you want and compute the above.

Remark 75.4.2 — You have seen a "free resolution" before in a disguised form — in Section 18.3, we proved the structure theorem of finitely-generated modules over PID by writing any module M as $R^{\oplus d}/K$, with both $R^{\oplus d}$ and K free. This gives a free resolution

$$\cdots \to 0 \to K \hookrightarrow R^{\oplus d} \twoheadrightarrow M \to 0.$$

Intuitively, you can think of the Ext functor as measuring the "maps that should be there but aren't" — you will gradually gain some intuitions after seeing some examples.[a]

[a] Taken from https://mathoverflow.net/a/679.

Lemma 75.4.3 (Computing the Ext functor)
For any abelian groups G, H, H' we have

(a) $\operatorname{Ext}(H \oplus H', G) = \operatorname{Ext}(H, G) \oplus \operatorname{Ext}(H', G)$.

(b) $\operatorname{Ext}(H, G) = 0$ for H free, and

(c) $\operatorname{Ext}(\mathbb{Z}/n\mathbb{Z}, G) = G/nG$.

Proof. For (a), note that if $\cdots \to F_1 \to F_0 \to H \to 0$ and $\cdots \to F_1' \to F_0' \to F_0' \to H' \to 0$ are free resolutions, then so is $F_1 \oplus F_1' \to F_0 \oplus F_0' \to H \oplus H' \to 0$.

For (b), note that $0 \to H \to H \to 0$ is a free resolution.

Part (c) follows by taking the free resolution

$$0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

and applying $\operatorname{Hom}(-, G)$ to it.

Question 75.4.4. Finish the proof of (c) from here. □

Question 75.4.5. Some Ext practice: compute $\operatorname{Ext}(\mathbb{Z}^{\oplus 2015}, G)$ and $\operatorname{Ext}(\mathbb{Z}/30\mathbb{Z}, \mathbb{Z}/4\mathbb{Z})$.

§75.5 Explanation for universal coefficient theorem

There is so much unexplained symbols and formulas in the previous chapter that may make you scream:

> I don't care if $\mathbb{CP}^2$ and $S^2 \vee S^4$ are distinct anymore! What are these spaces anyway?

Nevertheless, it is not all that difficult. There are two key points to be read from the theorem:

- Even though $H_n(A_\bullet) = 0$, it is still possible for $H^n(A_\bullet; G) \neq 0$ if $\operatorname{Ext}(H_{n-1}(A_\bullet), G) \neq 0$.

  In low-dimensional cases, we can actually visualize it — Section 75.7 does that for the Klein bottle.

- $H^n(A_\bullet; G)$ is uniquely determined by $H_n(A_\bullet)$ and G, regardless of what $A_\bullet$ is, as long as each A_n is free.

Which means: if you wish, you can forget about the formula in the universal coefficient theorem, and use the cellular chain complex $\operatorname{Cells}_\bullet(X)$ to compute cohomology by:

$$H^n(X; G) = \frac{\ker(\operatorname{Hom}(\operatorname{Cells}_n(X), G) \to \operatorname{Hom}(\operatorname{Cells}_{n+1}(X), G))}{\operatorname{im}(\operatorname{Hom}(\operatorname{Cells}_{n-1}(X), G) \to \operatorname{Hom}(\operatorname{Cells}_n(X), G))}.$$

After all, the cellular chain complex and the singular chain complex are both free and have the same homology groups, so by the universal coefficient theorem they must have the same cohomology groups.

Nevertheless, the formula of the universal coefficient theorem is desirable because, more often than not, the chain complex $A_\bullet$ is more complicated than $H_\bullet(A_\bullet)$.

Example 75.5.1

The Klein bottle's cellular chain complex has the following form:

$$\dots \to \mathbb{Z} \xrightarrow{1 \mapsto (0,2)} \mathbb{Z}^2 \xrightarrow{(a,b) \mapsto 0} \mathbb{Z}.$$

The homology groups is:

$$H_2 = 0, H_1 = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, H_0 = \mathbb{Z}.$$

It's indeed simpler, but only marginally (there are 3 generators instead of 4, and we don't need to keep track of the maps) because cellular homology is already so efficient.

Where does the formula come from, again? You can think of it like this. Because the universal coefficient theorem tells us that $H^\bullet(A_\bullet; G)$ only depends on $H_\bullet(A_\bullet)$, if we're given $H_\bullet$, we just construct *any* chain complex of free abelian groups $A_\bullet$ and dualize it.

Assume $H_k = 0$ for every terms, except $H_{n-1} \neq 0$. Then, tautologically, $H^n \cong \operatorname{Ext}(H_{n-1}; G)$ — a free resolution *is* a chain complex!

Exercise 75.5.2. Verify this. (Hint: Starting from the exact sequence $Z_{n-1} \to H_{n-1} \to 0$. Can you extend it to a free resolution of H_{n-1}?)

Assume $H_k = 0$ for every terms, except $H_n \neq 0$. Then we can see $H^n \cong \operatorname{Hom}(H_n, G)$.

The universal coefficient theorem simply states that the choice of free resolution doesn't matter, and that if the other terms can be nonzero, H^n is the direct sum of the two groups in the two cases above.

If you want, you can even prove the fact that the choice of free resolution does not matter yourself — it's a bit tricky, but not all that difficult. It boils down to the construction of maps between the chain complexes (it's not difficult to ensure the diagram commutes, the groups are free so we can send the basis wherever we want), and show the two free resolutions are chain homotopic.

§75.6 Example computation of cohomology groups

Prototypical example for this section: Possibly $H^n(S^m)$.

The universal coefficient theorem gives us a direct way to compute any cohomology groups, provided we know the homology ones.

Example 75.6.1 (Cohomology groups of S^m)
It is straightforward to compute $H^n(S^m)$ now: all the Ext terms vanish since $H_n(S^m)$ is always free, and hence we obtain that

$$H^n(S^m) \cong \operatorname{Hom}(H_n(S^m), G) \cong \begin{cases} G & n = m, n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Example 75.6.2 (Cohomology groups of torus)
This example has no nonzero Ext terms either, since this time $H^n(S^1 \times S^1)$ is always free. So we obtain

$$H^n(S^1 \times S^1) \cong \operatorname{Hom}(H_n(S^1 \times S^1), G).$$

Since $H_n(S^1 \times S^1)$ is $\mathbb{Z}$, $\mathbb{Z}^{\oplus 2}$, $\mathbb{Z}$ in dimensions $n = 1, 2, 1$ we derive that

$$H^n(S^1 \times S^1) \cong \begin{cases} G & n = 0, 2 \\ G^{\oplus 2} & n = 1. \end{cases}$$

From these examples one might notice that:

Lemma 75.6.3 (0th and 1th cohomology groups are just duals)
For $n = 0$ and $n = 1$, we have

$$H^n(X; G) \cong \operatorname{Hom}(H_n(X), G).$$

Proof. It's already been shown for $n = 0$. For $n = 1$, notice that $H_0(X)$ is free, so the Ext term vanishes. $\square$

Example 75.6.4 (Cohomology groups of Klein bottle)
This example will actually have Ext term. Recall from Example 74.5.4 that if K is a Klein Bottle then its homology groups are $\mathbb{Z}$ in dimension $n = 0$ and $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ in $n = 1$, and 0 elsewhere.
For $n = 0$, we again just have $H^0(K; G) \cong \operatorname{Hom}(\mathbb{Z}, G) \cong G$. For $n = 1$, the Ext

term is $\operatorname{Ext}(H_0(K), G) \cong \operatorname{Ext}(\mathbb{Z}, G) = 0$ so

$$H^1(K; G) \cong \operatorname{Hom}(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, G) \cong G \oplus \operatorname{Hom}(\mathbb{Z}/2\mathbb{Z}, G).$$

We have that $\operatorname{Hom}(\mathbb{Z}/2\mathbb{Z}, G)$ is the subgroup of elements of order 2 in G (and $0 \in G$).
But for $n = 2$, we have our first interesting Ext group: the exact sequence is

$$0 \to \operatorname{Ext}(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, G) \to H^2(X; G) \to \underbrace{H_2(X)}_{=0} \to 0.$$

Thus, we have

$$H^2(X; G) \cong (\operatorname{Ext}(\mathbb{Z}, G) \oplus \operatorname{Ext}(\mathbb{Z}/2\mathbb{Z}, G)) \oplus 0 \cong G/2G.$$

All the higher groups vanish. In summary:

$$H^n(X; G) \cong \begin{cases} G & n = 0 \\ G \oplus \operatorname{Hom}(\mathbb{Z}/2\mathbb{Z}, G) & n = 1 \\ G/2G & n = 2 \\ 0 & n \geq 3. \end{cases}$$

§75.7 Visualization of cohomology groups

We try to make sense of $C^n(X; G)$ and $H^n(X; G)$, for higher values of n.

As above, $C_n(X; G)$ is the free abelian group on n-simplices on X, so an element $f \in C^n(X; G)$ is a function that takes each n-simplex to an element of G (and extends linearly to all of $C_n(X; G)$).

This assignment of value need not have any nice properties — recall that a n-simplex is simply a (continuous) map $\sigma \colon \Delta^n \to X$, and different maps σ_1 and σ_2 are considered different even though $\operatorname{im} \sigma_1 = \operatorname{im} \sigma_2$. In particular,

- If $[v_0, v_1, v_2]$ is a singular simplex, it need not be the case that $f([v_0, v_1, v_2]) + f([v_0, v_2, v_1]) = 0$.
- A singular n-simplex ($n \geq 1$) with image contained in a point need not be mapped to 0 by f.

But it *does not matter* that elements of $C_n(X)$ aren't this nice! We will see below why this is the case.

In the homology case (Definition 71.2.2), we defined:

$$\begin{aligned} Z_n(X) &:= \ker \left(C_n(X) \xrightarrow{\partial} C_{n-1}(X) \right), \\ B_n(X) &:= \operatorname{im} \left(C_{n+1}(X) \xrightarrow{\partial} C_n(X) \right), \\ H_n(X) &:= Z_n(X)/B_n(X). \end{aligned}$$

Elements of $Z_n(X)$ and $B_n(X)$ are called cycles and boundaries respectively, with the obvious geometrical interpretation.

So,

$$H_n(X) = \frac{n\text{-cycles}}{n\text{-boundaries}}.$$

For the current section, we will temporarily define:

$$Z^n(X;G) \coloneqq \ker\left(C^n(X;G) \xrightarrow{\delta} C^{n+1}(X;G)\right),$$
$$B^n(X;G) \coloneqq \operatorname{im}\left(C^{n-1}(X;G) \xrightarrow{\delta} C^n(X;G)\right),$$
$$H^n(X;G) \coloneqq Z^n(X;G)/B^n(X;G).$$

For this section, we will call elements of $Z^n(X;G)$ the **cocycles** and elements of $B^n(X;G)$ the **coboundaries** respectively. Once again,

$$H^n(X;G) = \frac{n\text{-cocycles}}{n\text{-coboundaries}}.$$

It's less clear geometrically why the elements are named as above, but if we assume the group G is a *field* (where the group operation is the addition operation in the field), then[3] we have:

- a n-cocycle is a map that sends every n-boundary to $0 \in G$;
- a n-coboundary is a map that sends every n-cycle to $0 \in G$.

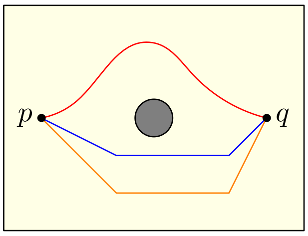
The first statement is clear (definition chasing), the second statement is only generally true in one direction (that a coboundary sends every cycle to 0; but a map that sends every cycle to 0 need not be a coboundary — we will see this later on with the Klein bottle example).

Let us see what a n-cocycle must look like. First,

Homotopic chains with the same boundary are mapped to the same value by cocycles.

We defined what it means for two k-simplices to be homotopic in Section 65.4 — in the current situation, we require in addition that the boundaries are always fixed.

For instance, the blue and the orange 1-simplices below are homotopic, but not the red 1-simplex.

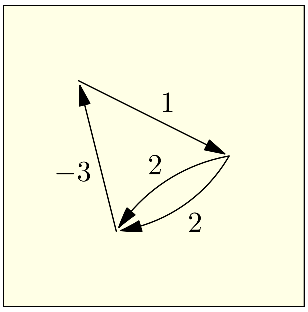


Proof is not difficult — you just need to show that the difference between two homotopic k-simplices is the boundary of something (their interior!), and write the interior as the sum of some $k+1$-simplices. (Hint: The easiest way is actually to write the interior as the difference of two $k+1$-simplices instead, and be careful of vertex ordering issues.)

Exercise 75.7.1. Finish the proof.

A typical 1-cocycle might look something like this, where each arrow is labeled with the value assigned to that 1-simplex. Remember that a cycle must be mapped to 0.

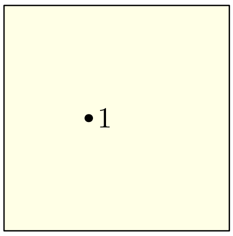
[3] Refer to https://math.stackexchange.com/q/4712676.



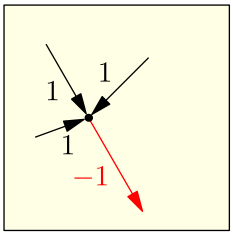
Now, the next observation is that:

If we only consider cocycles modulo coboundaries, we basically only care about values assigned to the cycles.

Why? Remember that a k-coboundary is the δ of some $(k-1)$-cochain. So, given this 0-cochain:



Its δ would look something like this:



So, roughly speaking,

By adding or subtracting a coboundary to a given cochain, we can adjust the value assigned to most chains however we want.

I said "most chains" because, if the chains form a *cycle*, adding a coboundary won't let us change its assigned value.

Fortunately,

- Cycles that are *boundaries* always get assigned the value 0.
- Homotopic cycles get assigned the same value.

  As a generalization, in fact, cycles that are homologous (i.e. they get mapped to the same value under the map $Z_k(X) \twoheadrightarrow H_k(X)$) are assigned the same value.

Therefore,

Knowing the value of a cocycle on each "cycle modulo boundary" almost determines that cocycle, modulo coboundaries.

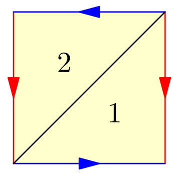
In symbols: $H^n(X; G)$ is "almost isomorphic" to $\operatorname{Hom}(H_n(X), G)$.

In other words, a cocycle modulo coboundary can be "evaluated" on a cycle modulo boundary.

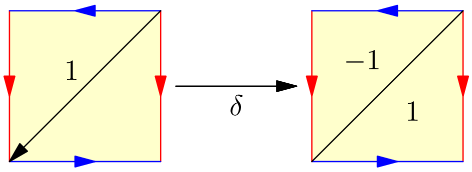
This is precisely what the universal coefficient theorem states, although it says something more: the "error term" is exactly $\operatorname{Ext}(H_{n-1}(X), G)$.

Why would the error term exist? We had an example above, computing $H^2(K; G)$ for K the Klein bottle. Let us work through it geometrically, assume $G = \mathbb{Z}$ for now.

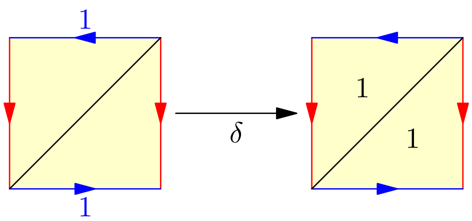
A typical 2-cochain $f \in C^2(K; \mathbb{Z})$ may look something like this. (Only value assigned to a few 2-simplices is depicted, there are too many 2-simplices for us to draw.)

$f =$ 

A coboundary may look like this — identical to the situation above, the value assigned to particular simplex doesn't matter, we can "transfer" the assigned value between the two simplices by adding a coboundary.



So, we may just say that the value assigned to the whole surface of the Klein bottle is 3 — formally, let $e^2_K \in C_2(K)$ be the sum of the two 2-simplices above, we can write $f(e^2_K) = 3$. However:

$g =$ 

The boundary of the 2-chain corresponding to the whole surface of the Klein bottle is 2 times the blue edge, so δ of the 1-cochain whose value on the blue edge is 1 will assign the value 2 to e^2_K.

In symbols: let $e^1_b \in C_1(K)$ be the blue edge, pick $g \in C^1(K; \mathbb{Z})$ such that $g(e^1_b) = 1$, then $\delta(g)(e^2_K) = 2$. Even though e^2_K is not a cycle, we still need to care about its assigned value modulo 2! Because adding or subtracting the coboundary $\delta(g)$ can only adjust its values in increments of 2.

Therefore,

If the region $e^k \in C_k(X)$ has a boundary $\partial e^k \in C_{k-1}(X)$ divisible by n, then we cares about the value assigned to e^k, modulo n.

This explains where the error term $\operatorname{Ext}(H_{n-1}(X), G)$ comes from.

We have another comparison with de Rham cohomology in Section 76.2 — in that case, the group G is a field, $\mathbb{R}$, so $\operatorname{Ext}(H_{n-1}(X), G)$ is always zero.

§75.8 Relative cohomology groups

One can also define relative cohomology groups in the obvious way: dualize the chain complex

$$\dots \xrightarrow{\partial} C_1(X, A) \xrightarrow{\partial} C_0(X, A) \to 0$$

to obtain a cochain complex

$$\dots \xleftarrow{\delta} C^1(X, A; G) \xleftarrow{\delta} C^0(X, A; G) \leftarrow 0.$$

We can take the cohomology groups of this.

Definition 75.8.1. The groups thus obtained are the **relative cohomology groups** are denoted $H^n(X, A; G)$.

In addition, we can define reduced cohomology groups as well. One way to do it is to take the augmented singular chain complex

$$\dots \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

and dualize it to obtain

$$\dots \xleftarrow{\delta} C^1(X; G) \xleftarrow{\delta} C^0(X; G) \xleftarrow{\varepsilon^\vee} \underbrace{\operatorname{Hom}(\mathbb{Z}, G)}_{\cong G} \leftarrow 0.$$

Since the $\mathbb{Z}$ we add is also free, the universal coefficient theorem still applies. So this will give us reduced cohomology groups.

However, since we already defined the relative cohomology groups, it is easiest to simply define:

Definition 75.8.2. The **reduced cohomology groups** of a nonempty space X, denoted $\widetilde{H}^n(X; G)$, are defined to be $H^n(X, \{*\}; G)$ for some point $* \in X$.

§75.9 A few harder problems to think about

Problem 75A* (Wedge product cohomology)**.** For any G and n we have

$$\widetilde{H}^n(X \vee Y; G) \cong \widetilde{H}^n(X; G) \oplus \widetilde{H}^n(Y; G).$$

Problem 75B†. Prove that for a field F of characteristic zero and a space X with finitely generated homology groups:

$$H^k(X, F) \cong (H_k(X))^\vee.$$

Thus over fields cohomology is the dual of homology.

Problem 75C ($\mathbb{Z}/2\mathbb{Z}$-cohomology of $\mathbb{RP}^n$)**.** Prove that

$$H^m(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & m = 0, \text{ or } m \text{ is odd and } m = n \\ \mathbb{Z}/2\mathbb{Z} & 0 < m < n \text{ and } m \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

76 Application of cohomology

In this final chapter on topology, I'll state (mostly without proof) some nice properties of cohomology groups, and in particular introduce the so-called cup product. For an actual treatise on the cup product, see [Ha02] or [Ma13a].

As mentioned in the previous chapter, you can put all the cohomology groups $H^\bullet(X)$ together to form the *cohomology ring*, which gives more structure than the case of homology — enough structure to allow distinguishing between $\mathbb{CP}^2$ and $S^2 \vee S^4$, or between $\mathbb{CP}^3$ and $S^2 \times S^4$.

Even though the description above is completely non-descriptive (it doesn't give you insight into *what* the structure is about), and actually, some people would say:

> It does not matter what homology measures intuitively, as it is a convenient tool that takes something very difficult (topology) and turns it into something simple (abelian group).

Nevertheless, it is interesting that the cup product *is actually visualizable*! At least when the dimension does not exceed 3.

§76.1 Poincaré duality

First cool result: you may have noticed symmetry in the (co)homology groups of "nice" spaces like the torus or S^n. In fact this is predicted by:

Theorem 76.1.1 (Poincaré duality)
If M is a smooth oriented compact n-manifold, then we have a natural isomorphism

$$H^k(M;\mathbb{Z}) \cong H_{n-k}(M)$$

for every k. In particular, $H^k(M) = 0$ for $k > n$.

So for smooth oriented compact manifolds, cohomology and homology groups are not so different.

From this follows the symmetry that we mentioned when we first defined the Betti numbers:

Corollary 76.1.2 (Symmetry of Betti numbers)
Let M be a smooth oriented compact n-manifold, and let b_k denote its Betti number. Then

$$b_k = b_{n-k}.$$

Proof. Problem 76A†. □

§76.2 de Rham cohomology

We now reveal the connection between differential forms and singular cohomology.

Let M be a smooth manifold. We are interested in the homology and cohomology groups of M. We specialize to the case $G = \mathbb{R}$, the additive group of real numbers.

> **Question 76.2.1.** Check that $\operatorname{Ext}(H, \mathbb{R}) = 0$ for any finitely generated abelian group H.

Thus, with real coefficients the universal coefficient theorem says that

$$H^k(M; \mathbb{R}) \cong \operatorname{Hom}(H_k(M), \mathbb{R}) = (H_k(M))^\vee$$

where we view $H_k(X)$ as a real vector space. So, we'd like to get a handle on either $H_k(M)$ or $H^k(M; \mathbb{R})$.

Consider the cochain complex

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M) \xrightarrow{d} \dots$$

and let $H^k_{\mathrm{dR}}(M)$ denote its cohomology groups. Thus the de Rham cohomology is the closed forms modulo the exact forms.

$$\text{Cochain} : \text{Cocycle} : \text{Coboundary} = k\text{-form} : \text{Closed form} : \text{Exact form}.$$

The whole punch line is:

Theorem 76.2.2 (de Rham's theorem)

For any smooth manifold M, we have a natural isomorphism

$$H^k(M; \mathbb{R}) \cong H^k_{\mathrm{dR}}(M).$$

So the theorem is that the real cohomology groups of manifolds M are actually just given by the behavior of differential forms. Thus,

One can metaphorically think of elements of cohomology groups as G-valued differential forms on the space.

Why does this happen? In fact, we observed already behavior of differential forms which reflects holes in the space. For example, let $M = S^1$ be a circle and consider the **angle form** α (see Example 44.7.4). The form α is closed, but not exact, because it is possible to run a full circle around S^1. So the failure of α to be exact is signaling that $H_1(S^1) \cong \mathbb{Z}$.

As another piece of intuition, note that:

- each k-differential form ω can be interpreted as a function that takes each k-smooth submanifold $S \subseteq M$, and returns a real number $\int_S \omega$.
- let us pretend that all k-simplices are smooth for now. Then we have:
  - The k-cochains are the functions that sends each k-simplex to a real number.
  - The k-cocycles are the k-cochains that sends the boundaries to 0.
  - The k-coboundaries are the k-cochains that sends the cycles to 0.

  Meanwhile:
  - The differential forms are the functions that sends each k-simplex to a real number, satisfying certain linearity and smoothness properties — for instance:

  * if $k \geq 1$ and a k-simplex has the image contained in a point, then it must be sent to 0;
  * if we reparametrize a k-simplex, the assigned value must be the same;
  * if we flip two vertices of a k-simplex, the assigned value must be negated;
  * if a k-simplex can be formed by gluing two k-simplices along a face, then the assigned value must be the sum of the corresponding values assigned to the sub-k-simplices;
  * etc.
- The closed forms are the differential forms that sends the boundaries to 0;
- The exact forms are the differential forms that send the cycles to 0.

We can't help but noticing the parallel — the point is:

$$H^k(M;\mathbb{R}) = \frac{\text{cocycles}}{\text{coboundaries}} \cong \frac{\text{cocycles} \cap \text{differential forms}}{\text{coboundaries} \cap \text{differential forms}} = H^k_{\text{dR}}(M).$$

Roughly speaking, both the numerator and the denominator on the left are bigger, and they *cancels out*. We can compare this with Section 74.3.

Or, as a figure (for space reasons, the group of differential forms is denoted D):

$$\begin{tikzcd}
D & & \text{cocycles} & \\
& \text{cocycles} \cap D \arrow[ul] \arrow[ur] & & \text{coboundaries} \arrow[ul] \\
& & \text{coboundaries} \cap D \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

This is precisely the setup of the second isomorphism theorem,[1] and you can try to work out why the two quotients are isomorphic.

§76.3 Graded rings

Prototypical example for this section: Polynomial rings are commutative graded rings, while $\bigwedge^\bullet(V)$ is anticommutative.

In the de Rham cohomology, the differential forms can interact in another way: given a k-form α and an ℓ-form β, we can consider a $(k+\ell)$-form

$$\alpha \wedge \beta.$$

So we can equip the set of forms with a "product", satisfying $\beta \wedge \alpha = (-1)^{k\ell}\alpha \wedge \beta$. This is a special case of a more general structure:

Definition 76.3.1. A **graded pseudo-ring** R is an abelian group

$$R = \bigoplus_{d \geq 0} R^d$$

where R^0, R^1, ..., are abelian groups, with an additional associative binary operation $\times \colon R \to R$. We require that if $r \in R^d$ and $s \in R^e$, we have $rs \in R^{d+e}$. Elements of an R^d are called **homogeneous elements**; if $r \in R^d$ and $r \neq 0$, we write $|r| = d$.

[1]See Section 3.6.

Note that we do *not* assume commutativity. In fact, these "rings" may not even have an identity 1. We use other words if there are additional properties:

Definition 76.3.2. A **graded ring** is a graded pseudo-ring with 1. If it is commutative we say it is a **commutative graded ring**.

Definition 76.3.3. A graded (pseudo-)ring R is **anticommutative** if for any homogeneous r and s we have

$$rs = (-1)^{|r||s|} sr.$$

Remark 76.3.4 — Why not $rs = -sr$? This definition is inspired by the fact that the wedge product is anticommutative. Note that, for $f_1, \dots, f_r, g_1, \dots, g_s$ being 0-forms, let $f = df_1 \wedge df_2 \wedge \dots \wedge df_r$ be a r-form and $g = dg_1 \wedge dg_2 \wedge \dots \wedge dg_s$ be a s-form, then starting from the expression

$$f \wedge g = (df_1 \wedge df_2 \wedge \dots \wedge df_r) \wedge (dg_1 \wedge dg_2 \wedge \dots \wedge dg_s)$$

if you repeatedly swap two adjacent entries, it will take rs swaps total in order to obtain the expression

$$g \wedge f = (dg_1 \wedge dg_2 \wedge \dots \wedge dg_s) \wedge (df_1 \wedge df_2 \wedge \dots \wedge df_r).$$

By linearity, we can prove that in general, for any r-form f and any s-form g, we have $fg = (-1)^{rs} gf$.

To summarize:

Flavors of graded rings	Need not have 1	Must have a 1
No Assumption	graded pseudo-ring	graded ring
Anticommutative	anticommutative pseudo-ring	anticommutative ring
Commutative		commutative graded ring

Example 76.3.5 (Examples of graded rings)

(a) The ring $R = \mathbb{Z}[x]$ is a **commutative graded ring**, with the dth component being the multiples of x^d.

(b) The ring $R = \mathbb{Z}[x, y, z]$ is a **commutative graded ring**, with the dth component being the abelian group of homogeneous degree d polynomials (and 0).

(c) Let V be a vector space, and consider the abelian group

$$\bigwedge^{\bullet}(V) = \bigoplus_{d \ge 0} \bigwedge^{d}(V).$$

For example, $e_1 + (e_2 \wedge e_3) \in \bigwedge^{\bullet}(V)$, say. We endow $\bigwedge^{\bullet}(V)$ with the product $\wedge$, which makes it into an **anticommutative ring**.

(d) Consider the set of differential forms of a manifold M, say

$$\Omega^{\bullet}(M) = \bigoplus_{d \ge 0} \Omega^{d}(M)$$

endowed with the product $\wedge$. This is an **anticommutative ring**.

All four examples have a multiplicative identity.

Let's return to the situation of $\Omega^\bullet(M)$. Consider again the de Rham cohomology groups $H^k_{\text{dR}}(M)$, whose elements are closed forms modulo exact forms. We claim that:

Lemma 76.3.6 (Wedge product respects de Rham cohomology)
The wedge product induces a map

$$\wedge\colon H^k_{\text{dR}}(M) \times H^\ell_{\text{dR}}(M) \to H^{k+\ell}_{\text{dR}}(M).$$

Proof. First, we recall that the operator d satisfies

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + \alpha \wedge (d\beta).$$

Now suppose α and β are closed forms. Then from the above, $\alpha \wedge \beta$ is clearly closed. Also if α is closed and $\beta = d\omega$ is exact, then $\alpha \wedge \beta$ is exact, from the identity

$$d(\alpha \wedge \omega) = d\alpha \wedge \omega + \alpha \wedge d\omega = \alpha \wedge \beta.$$

Similarly if α is exact and β is closed then $\alpha \wedge \beta$ is exact. Thus it makes sense to take the product modulo exact forms, giving the theorem above. $\square$

Therefore, we can obtain a *anticommutative ring*

$$H^\bullet_{\text{dR}}(M) = \bigoplus_{k \ge 0} H^k_{\text{dR}}(M)$$

with $\wedge$ as a product, and $1 \in \bigwedge^0(\mathbb{R}) = \mathbb{R}$ as the identity.

§76.4 Cup products

Inspired by this, we want to see if we can construct a similar product on $\bigoplus_{k \ge 0} H^k(X; R)$ for any topological space X and ring R (where R is commutative with 1 as always). The way to do this is via the *cup product*.

Then this gives us a way to multiply two cochains, as follows.

Definition 76.4.1. Suppose $\phi \in C^k(X; R)$ and $\psi \in C^\ell(X; R)$. Then we can define their **cup product** $\phi \smile \psi \in C^{k+\ell}(X; R)$ to be

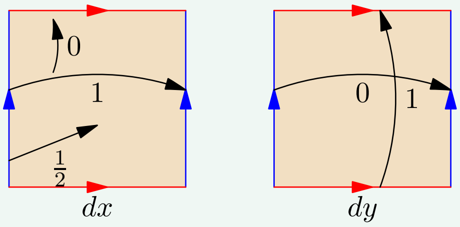
$$(\phi \smile \psi)([v_0, \dots, v_{k+\ell}]) = \phi([v_0, \dots, v_k]) \cdot \psi([v_k, \dots, v_{k+\ell}])$$

where the multiplication is in R.

Question 76.4.2. Assuming R has a 1, which 0-cochain is the identity for $\smile$?

Remark 76.4.3 (Warning) — While you can interpret a n-differential form as a n-cochain the obvious way, the cup product is *not* directly a generalization of the wedge product! For example, let $X = \mathbb{R}^2$, and try to evaluate $dx \smile dy$ on $[v_0, v_1, v_2]$ and $[v_2, v_1, v_0]$ where $v_0 = (1, 0)$, $v_1 = (0, 0)$, $v_2 = (0, 1)$, assume all of the edges are straight lines.
This is because we are not having the alternation operator. Refer to Section 44.5 for details. In this case, the ring G might be $\mathbb{Z}$ where not all nonzero elements have

an inverse, so division would cause trouble.
Nevertheless, the differences will nicely cancel out, and we still have the corresponding element in the cohomology group equal to the element interpreted by the wedge product $dx \wedge dy$ — this is what we mean by $H^\bullet(M;\mathbb{R}) \cong H^\bullet_{\mathrm{dR}}(M)$, stated below.
Let us consider the familiar example of a torus, and the 1-cocycles "dx" and "dy".



From what we know about the wedge product, we want $(dx \wedge dy)(T) = 1$ for T the whole torus (up to a $\pm$ sign). Indeed, with the definition above (work it out! Divide T into two triangles arbitrarily) it will work.
Nevertheless, we don't really care about the cup product itself as much as the induced cup product on the homology ring.

First, we prove an analogous result as before:

Lemma 76.4.4 (δ with cup products)
We have $\delta(\phi \smile \psi) = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi$.

Proof. Direct $\sum$ computations. $\square$

Thus, by the same routine we used for de Rham cohomology, we get an induced map

$$\smile \colon H^k(X;R) \times H^\ell(X;R) \to H^{k+\ell}(X;R).$$

We then define the **singular cohomology ring** whose elements are finite sums in

$$H^\bullet(X;R) = \bigoplus_{k \ge 0} H^k(X;R)$$

and with multiplication given by $\smile$. Thus it is a graded ring (with $1_R \in R$ the identity) and is in fact anticommutative:

Proposition 76.4.5 (Cohomology is anticommutative)
$H^\bullet(X;R)$ is an anticommutative ring, meaning $\phi \smile \psi = (-1)^{k\ell} \psi \smile \phi$.

For a proof, see [Ha02, Theorem 3.11, pages 210-212]. Moreover, we have the de Rham isomorphism

Theorem 76.4.6 (de Rham extends to ring isomorphism)

For any smooth manifold M, the isomorphism of de Rham cohomology groups to singular cohomology groups in facts gives an isomorphism

$$H^\bullet(M;\mathbb{R}) \cong H^\bullet_{\mathrm{dR}}(M)$$

of anticommutative rings.

Therefore, if "differential forms" are the way to visualize the elements of a cohomology group, the wedge product is the correct way to visualize the cup product.

We now present (mostly without proof) the cohomology rings of some common spaces.

Example 76.4.7 (Cohomology of torus)

The cohomology ring $H^\bullet(S^1 \times S^1;\mathbb{Z})$ of the torus is generated by elements $|\alpha| = |\beta| = 1$ which satisfy the relations $\alpha \smile \alpha = \beta \smile \beta = 0$, and $\alpha \smile \beta = -\beta \smile \alpha$. (It also includes an identity 1.) Thus as a $\mathbb{Z}$-module it is

$$H^\bullet(S^1 \times S^1;\mathbb{Z}) \cong \mathbb{Z} \oplus [\alpha\mathbb{Z} \oplus \beta\mathbb{Z}] \oplus (\alpha \smile \beta)\mathbb{Z}.$$

This gives the expected dimensions $1 + 2 + 1 = 4$. It is anti-commutative.

You have already seen the elements α and β as the elements called dx and dy in the remark above.

Example 76.4.8 (Cohomology ring of S^n)

Consider S^n for $n \geq 1$. The nontrivial cohomology groups are given by $H^0(S^n;\mathbb{Z}) \cong H^n(S^n;\mathbb{Z}) \cong \mathbb{Z}$. So as an abelian group

$$H^\bullet(S^n;\mathbb{Z}) \cong \mathbb{Z} \oplus \alpha\mathbb{Z}$$

where α is the generator of $H^n(S^n, \mathbb{Z})$.

Now, observe that $|\alpha \smile \alpha| = 2n$, but since $H^{2n}(S^n;\mathbb{Z}) = 0$ we must have $\alpha \smile \alpha = 0$. So even more succinctly,

$$H^\bullet(S^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^2).$$

Confusingly enough, this graded ring is both commutative *and* anti-commutative. The reason is that $\alpha \smile \alpha = 0 = -(\alpha \smile \alpha)$.

Example 76.4.9 (Cohomology ring of real and complex projective space)

It turns out that

$$H^\bullet(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\alpha]/(\alpha^{n+1})$$
$$H^\bullet(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[\beta]/(\beta^{n+1})$$

where $|\alpha| = 1$ is a generator of $H^1(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$ and $|\beta| = 2$ is a generator of $H^2(\mathbb{CP}^n;\mathbb{Z})$.

Confusingly enough, both graded rings are commutative *and* anti-commutative. In the first case it is because we work in $\mathbb{Z}/2\mathbb{Z}$, for which $1 = -1$, so anticommutative is actually equivalent to commutative. In the second case, all nonzero homogeneous elements have degree 2.

Already we have an interesting example where the cup product $\smile$ is different from the wedge product $\wedge$ — if $n \geq 2$, then the generators α and β above has $\alpha \smile \alpha \neq 0$ and $\beta \smile \beta \neq 0$.

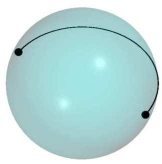
Let us try to see what happens here. The formula above says

$$H^\bullet(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\alpha]/(\alpha^3)$$

As an abelian group, there is a single nonzero element in $H^0(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z})$, $H^1(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z})$, and $H^2(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z})$, and the remaining groups are 0.

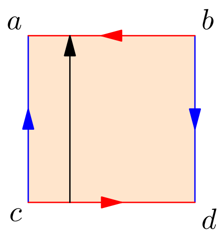
$\mathbb{RP}^2$ isn't too hard to visualize — it's just a 2-sphere, quotient by the relation to identify opposite vertices.

There is a 1-cycle on it that is not homologous to 0:



It's not very easy to show, but every such 1-cycle is homologous to each other, and double of that cycle is homologous to 0.

As such, $H^1(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) \cong \operatorname{Hom}(H_1(\mathbb{RP}^2), \mathbb{Z}/2\mathbb{Z})$, its only nontrivial element α maps each such 1-cycle to 1.



Consider $\alpha \smile \alpha$. Notice that α acts like both dx and dy at the same time (both the blue edge and the red edge got assigned the value 1), so it assigns the value 1 to the whole surface of the real projective plane! Thus it's nontrivial.

Exercise 76.4.10. Manually compute the cup product $\alpha \smile \alpha$ to verify that. (Divide the surface into some triangles. $[a, c, d] + [a, d, b] - [c, c, d] + [c, d, c]$ is a working choice. Verify that the boundary is nonzero, but is divisible by 2.)

§76.5 Relative cohomology pseudo-rings

For $A \subseteq X$, one can also define a relative cup product

$$H^k(X, A; R) \times H^\ell(X, A; R) \to H^{k+\ell}(X, A; R).$$

After all, if either cochain vanishes on chains in A, then so does their cup product. This lets us define **relative cohomology pseudo-ring** and **reduced cohomology**

pseudo-ring (by $A = \{*\}$), say

$$H^\bullet(X, A; R) = \bigoplus_{k \ge 0} H^k(X, A; R)$$
$$\widetilde{H}^\bullet(X; R) = \bigoplus_{k \ge 0} \widetilde{H}^k(X; R).$$

These are both **anticommutative pseudo-rings**. Indeed, often we have $\widetilde{H}^0(X; R) = 0$ and thus there is no identity at all.

Once again we have functoriality:

Theorem 76.5.1 (Cohomology (pseudo-)rings are functorial)
Fix a ring R (commutative with 1). Then we have functors

$$H^\bullet(-; R) : \mathsf{hTop}^{\mathrm{op}} \to \mathsf{GradedRings}$$
$$H^\bullet(-, -; R) : \mathsf{hPairTop}^{\mathrm{op}} \to \mathsf{GradedPseudoRings}.$$

Unfortunately, unlike with (co)homology groups, it is a nontrivial task to determine[2] the cup product for even nice spaces like CW complexes. So we will not do much in the way of computation. However, there is a little progress we can make.

§76.6 Wedge sums

Our goal is to now compute $\widetilde{H}^\bullet(X \vee Y)$. To do this, we need to define the product of two graded pseudo-rings:

Definition 76.6.1. Let R and S be two graded pseudo-rings. The **product pseudo-ring** $R \times S$ is the graded pseudo-ring defined by taking the underlying abelian group as

$$R \oplus S = \bigoplus_{d \ge 0} (R^d \oplus S^d).$$

Multiplication comes from R and S, followed by declaring $r \cdot s = 0$ for $r \in R$, $s \in S$.

Note that this is just graded version of the product ring defined in Example 4.3.8.

Exercise 76.6.2. Show that if R and S are graded rings (meaning they have 1_R and 1_S), then so is $R \times S$.

Now, the theorem is that:

Theorem 76.6.3 (Cohomology pseudo-rings of wedge sums)
We have

$$\widetilde{H}^\bullet(X \vee Y; R) \cong \widetilde{H}^\bullet(X; R) \times \widetilde{H}^\bullet(Y; R)$$

as graded pseudo-rings.

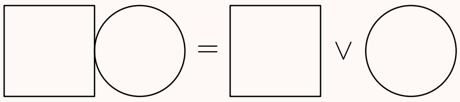
[2] Apart from the method of passing to differential form and back, that is. You have already computed a wedge product above.

Knowing just that the rings are isomorphic doesn't help much, it would be much better if you know what the isomorphism is — so that in simple cases, you can see for yourself the rings are isomorphic.

The isomorphism is the most trivial one: Given $f \in C^\bullet(X \vee Y; R)$ that assigns to each chain c inside $X \vee Y$ a value $f(c) \in R$, we can interpret it as an element of $C^\bullet(X)$, because each chain inside X is trivially a chain inside $X \vee Y$ that can be fed into f — formally, the embedding $X \hookrightarrow X \vee Y$ induces $C_\bullet(X) \hookrightarrow C_\bullet(X \vee Y)$. The map induces a $\widetilde{H}^\bullet(X \vee Y; R) \to \widetilde{H}^\bullet(X; R) \times \widetilde{H}^\bullet(Y; R)$, and it respects the ring multiplication i.e. the cup product.

Example 76.6.4

Let X and Y be depicted as in the following figure.



Let $f \in \widetilde{H}^1(X; \mathbb{Z})$ assigns $f(X) = 2$ to the whole square, and $g \in \widetilde{H}^1(Y; \mathbb{Z})$ assigns $g(Y) = 3$ to the whole circle. Then, of course the element corresponds to (f, g) inside $\widetilde{H}^1(X \vee Y)$ would assigns $2 + 3 = 5$ to the cocycle corresponding to the whole space $X \vee Y$.

This allows us to resolve the first question posed at the beginning. Let $X = \mathbb{CP}^2$ and $Y = S^2 \vee S^4$. We have that

$$H^\bullet(\mathbb{CP}^2; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^3).$$

Hence this is a graded ring generated by there elements:

- 1, in dimension 0.
- α, in dimension 2.
- α^2, in dimension 4.

Next, consider the reduced cohomology pseudo-ring

$$\widetilde{H}^\bullet(S^2 \vee S^4; \mathbb{Z}) \cong \widetilde{H}^\bullet(S^2; \mathbb{Z}) \oplus \widetilde{H}^\bullet(S^4; \mathbb{Z}).$$

Thus the absolute cohomology ring $H^\bullet(S^2 \vee S^4; \mathbb{Z})$ is a graded ring also generated by three elements.

- 1, in dimension 0 (once we add back in the 0th dimension).
- a_2, in dimension 2 (from $H^\bullet(S^2; \mathbb{Z})$).
- a_4, in dimension 4 (from $H^\bullet(S^4; \mathbb{Z})$).

Each graded component is isomorphic, like we expected. However, in the former, the product of two degree 2 generators is

$$\alpha \cdot \alpha = \alpha^2.$$

In the latter, the product of two degree 2 generators is

$$a_2 \cdot a_2 = a_2^2 = 0$$

since $a_2 \smile a_2 = 0 \in H^\bullet(S^2; \mathbb{Z})$.

Thus $S^2 \vee S^4$ and $\mathbb{CP}^2$ are not homotopy equivalent.

Intuitively, what the proof above says is:

The nontrivial 4-cocycle $a_4 \in H^4(S^2 \vee S^4; \mathbb{Z})$ has nothing to do with the 2-cocycle a_2, while the 4-cocycle $\alpha^2 \in H^4(\mathbb{CP}^2)$ is the cup product $\alpha \smile \alpha$ of the 2-cocycle α with itself.

The exercise below would be much easier to visualize, apart from the fact that $\mathbb{RP}^2$ is nonorientable — in fact, we have already seen above why $\alpha \smile \alpha \neq 0$ for the nonzero element $\alpha \in H^1(\mathbb{RP}^2)$.

> **Exercise 76.6.5.** Similarly, show that $S^1 \vee S^2$ and $\mathbb{RP}^2$ are not homotopy equivalent by showing $\widetilde{H}^\bullet(S^1 \vee S^2; \mathbb{Z}/2\mathbb{Z}) \ncong \widetilde{H}^\bullet(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z})$, even though each graded component is isomorphic.

§76.7 Cross product

In this section, we will define the cross product.

§76.7.i Motivation

Roughly speaking, the motivation is the following:

If X has a m-dimensional hole and Y has a n-dimensional hole, then $X \times Y$ has a $(m+n)$-dimensional hole.

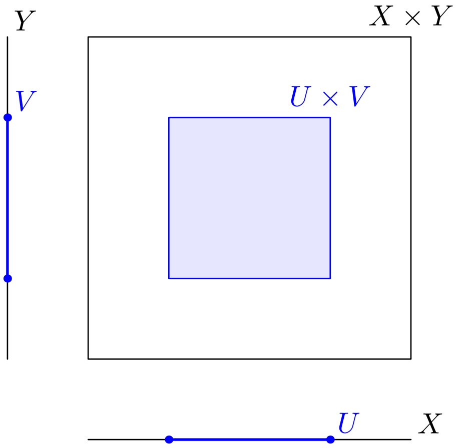
Which is true in most common cases under suitable interpretation of "holes" (either with homology, or with cohomology).

We will formalize and prove the statement above.

§76.7.ii Cross product on singular homology

First, we define the **cross product**, that takes a m-simplex $f \colon \Delta^m \to X$ and a n-simplex $g \colon \Delta^n \to Y$, and returns a $(m+n)$-chain $f \times g \in C_{m+n}(X \times Y)$.[3] This is really the most natural way you might define it: intuitively, the product of a m-dimensional cube in X and a n-dimensional cube in Y is a $(m+n)$-dimensional cube in $X \times Y$.

[3] As far as I know, this is just because the symbol $\times$ is a cross, and it has nothing to do with the cross product of vectors in $\mathbb{R}^3$.



In the case of a simplex, we need to subdivide $\Delta^m \times \Delta^n$ into finitely many copies of Δ^{m+n}.

If $n = 1$, we have already seen a subdivision when we worked with the prism operator. For the general case, refer to [**Ha02**, page 277] — the number blows up quickly, for example, you need $\binom{30}{15} = 155117520$ simplices to cover $\Delta^{15} \times \Delta^{15}$!

Formally, we can define the cross product of chains: that is, a function

$$C_m(X) \times C_n(Y) \xrightarrow{\times} C_{m \times n}(X \times Y).$$

We can prove that this induces a map on homology groups:

$$H_m(X) \times H_n(Y) \xrightarrow{\times} H_{m \times n}(X \times Y).$$

> **Exercise 76.7.1.** Let $X = Y = S^1$, so that $X \times Y$ is a torus. Let α be a generator of $H_1(X)$, and β be a generator of $H_1(Y)$. Show that $\alpha \times \beta$ is the generator of $H_2(X \times Y)$.

Actually, we have the following:

Theorem 76.7.2

If X and Y are CW complexes and R is a PID, then the cross product of two nonzero elements in $H_m(X)$ and $H_n(Y)$ is nonzero.

Thus formalize our intuition earlier — at least, if we use homology as a measure of "holes".

§76.7.iii Cross product is not a $\mathbb{Z}$-module homomorphism

For this section, if a and b are elements of the $\mathbb{Z}$-module $C_m(X)$ and $C_n(Y)$ respectively, we write $\times(a, b)$ to mean $a \times b \in C_{m+n}(X \times Y)$, and (a, b) to be the element that corresponds in the product $C_m(X) \times C_n(Y)$.

There is a little technical detail that we need to sort out — above, we writes

$$\times : C_m(X) \times C_n(Y) \to C_{m+n}(X \times Y).$$

But written this way, $\times$ is not a $\mathbb{Z}$-module homomorphism!

Example 76.7.3

Let a and b be any nonzero elements in $C_m(X)$ and $C_n(Y)$ respectively. Then,

$$\begin{aligned} \times(a,b) &= a \times b \\ 2 \cdot (a,b) &= (2a,2b) \\ \times(2 \cdot (a,b)) &= 4(a \times b). \end{aligned}$$

If we want to talk about isomorphism, or do anything with the $\mathbb{Z}$-module structure of $C_{m+n}(X \times Y)$ or $H_{m+n}(X \times Y)$, we'd better having a $\mathbb{Z}$-module homomorphism.

This is easy enough to fix: $\times$ is bilinear, so it's natural to consider the tensor product:

$$\times \colon C_m(X) \otimes_{\mathbb{Z}} C_n(Y) \to C_{m+n}(X \times Y).$$

With this notation, $\times(a \otimes b) = a \times b$. (As a side effect, we can also write $\times(a \otimes b + c \otimes d) = a \times b + c \times d$ now.)

And so, let us restate Theorem 76.7.2:

Theorem 76.7.4

If X and Y are CW complexes, then

$$\times \colon H_m(X) \otimes_{\mathbb{Z}} H_n(Y) \to H_{m+n}(X \times Y)$$

is an injective $\mathbb{Z}$-module homomorphism.

§76.7.iv Cross product on cellular homology

The definition with singular homology is quite clumsy — because we use simplices as the building blocks for the chains, the product of two simplices in X and Y becomes a huge collection of simplices in $X \times Y$.

We will now redefine the cross product using cellular homology — it can be safely skipped, since both definitions of the cross product gives identical result on the homology groups.

If X and Y are CW complexes, we can do better. We see that $X \times Y$ has a natural CW complex structure: for each cell e^m of X and cell e^n of Y, their product makes for a cell e^{m+n} of $X \times Y$.

Example 76.7.5

If X and Y are both line segments built from two 0-cells and one 1-cell, then their product $X \times Y$ has a natural CW complex structure containing:

- 4 0-cells,
- 4 1-cells,
- 1 2-cell.

Recall the cellular groups $\operatorname{Cells}_\bullet(X)$ from Chapter 74, each basis element corresponds to a cell in X. Then, we can define the cross product on the basis elements:

$$\times\colon \operatorname{Cells}_m(X) \otimes_{\mathbb{Z}} \operatorname{Cells}_n(Y) \to \operatorname{Cells}_{m+n}(X \times Y).$$

To be painfully explicit: let $e^m \in \operatorname{Cells}_m(X)$, $e^n \in \operatorname{Cells}_m(Y)$, then the cross product is defined by $e^m \times e^n = e^m \times e^n \in \operatorname{Cells}_{m+n}(X \times Y)$ — even the notation used is trivial.

Of course, this induces a map on the homology groups:

$$\times\colon H_m(X) \otimes_{\mathbb{Z}} H_n(Y) \to H_{m+n}(X \times Y).$$

This map is the same as the map we defined earlier.

§76.7.v Cross product on cellular cohomology

We do the same thing as above, but this time with cohomology — remember that homology and cohomology are slightly different measures of "holes", for K the Klein bottle then $H_2(X) = 0$ but $H^2(X;\mathbb{Z}) \neq 0$.

Given two cellular cochains $f \in \operatorname{Hom}(\operatorname{Cells}_m(X); R)$ and $g \in \operatorname{Hom}(\operatorname{Cells}_n(Y); R)$, we want to obtain a cochain $f \times g \in \operatorname{Hom}(\operatorname{Cells}_{m+n}(X \times Y); R)$.

Of course, it is defined in the most natural way possible: for a cell e^m of X and a cell e^n of Y, we have $(f \times g)(e^m \times e^n) = f(e^m) \cdot g(e^n)$.

Sounds good? Not yet — since not all $(m+n)$-cells e^{m+n} of $X \times Y$ is formed as a product of a m-cell in X and a n-cell in Y. For those, we simply declare that $(f \times g)(e^{m+n}) = 0$.

As usual, this map induces a R-module homomorphism on the cohomology groups:

$$\times\colon H^m(X;R) \otimes_R H^n(Y;R) \to H^{m+n}(X \times Y;R).$$

§76.7.vi Motivation: cross product of differential forms

The definition of the cross product of two cellular cochains above are clean, but may appear to be dry and unmotivated.

Turns out you can do the same thing on differential form. What's more, it gives a clean way of defining the wedge product $\alpha \wedge \beta$! Let's see it in action.

Instead of the definition, here are a few examples. Motivated readers may try to define the concept formally.

Example 76.7.6 (Examples of cross product of differential form)
Here are a few examples.

- If X and Y are the x-axis and the y-axis of the plane respectively, the cross product $dx \times 2dy$ is equal to $2(dx \wedge dy)$.

  Certainly this is natural — as dx assigns the value 1 to the vector $\mathbf{e}_1$, and $2dy$ assigns the value 2 to the vector $\mathbf{e}_2$, we get that $dx \times 2dy$ should assigns the value $1 \cdot 2 = 2$ to the unit square spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$ — that is, $\mathbf{e}_1 \wedge \mathbf{e}_2$.

- Let X be the xy-plane, and let Y be the z-axis. Consider the cross product $dx \times dz$. What 2-form should the result be?

  Certainly, we should have $(dx \times dz)(\mathbf{e}_1 \wedge \mathbf{e}_3) = 1$ and $(dx \times dz)(\mathbf{e}_2 \wedge \mathbf{e}_3) = 0$. But this isn't enough to uniquely determines $dx \times dz$.

  And so, we declares: $(dx \times dz)(\mathbf{e}_1 \wedge \mathbf{e}_2) = 0$. With this, we get $dx \times dz = dx \wedge dz$.

More generally, we can define the cross product by picking a basis for X and Y, and define the value of $\alpha \times \beta$ on the basis elements.

As promised — you can define the wedge product using the cross product. There's only one thing you can do:

Definition 76.7.7 (Definition of wedge product using the cross product)**.** For X a $\mathbb{R}$-vector space, let $\alpha \in (\bigwedge^m(X))^\vee$ and $\beta \in (\bigwedge^n(X))^\vee$, then $\alpha \wedge \beta \in (\bigwedge^{m+n}(X)$ is defined by

$$\alpha \wedge \beta = \Delta^*(\alpha \times \beta)$$

where $\Delta \colon X \to X \times X$, $\Delta(x) = (x, x)$ is the diagonal map. Recall that Δ^* denotes the pullback operation.

In simpler terms: to evaluate $\alpha \wedge \beta$ on a $(m+n)$-wedge in X, push it to $X \times X$ using the diagonal map, and give it to $\alpha \times \beta$.

§76.7.vii Piecing the cohomology groups together

Recall that we have above the R-module homomorphism

$$\times \colon H^m(X; R) \otimes_R H^n(Y; R) \to H^{m+n}(X \times Y; R).$$

We know that it is in fact possible to piece all the $H^\bullet(X; R)$ together to form an anticommutative graded ring, the cohomology ring. So we wish to extend the map to a R-algebra homomorphism

$$\times \colon H^\bullet(X; R) \otimes_R H^\bullet(Y; R) \to H^\bullet(X \times Y; R).$$

We haven't defined what the tensor product of two graded rings is yet — we will formally do that in the next section, but intuitively, it consists of all the $H^m(X; R) \otimes_R H^n(Y; R)$ pieced together.

§76.8 Künneth formula

We now wish to tell apart the spaces $S^2 \times S^4$ and $\mathbb{CP}^3$. In order to do this, we will need a formula for $H^n(X \times Y; R)$ in terms of $H^n(X; R)$ and $H^n(Y; R)$. These formulas are called **Künneth formulas**. In this section we will only use a very special case, which involves the tensor product of two graded rings.

Definition 76.8.1. Let A and B be two graded rings which are also R-modules (where R is a commutative ring with 1). We define the **tensor product** $A \otimes_R B$ as follows. As an abelian group, it is

$$A \otimes_R B = \bigoplus_{d \ge 0} \left(\bigoplus_{k=0}^{d} A^k \otimes_R B^{d-k} \right).$$

The multiplication is given on basis elements by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2).$$

Of course the multiplicative identity is $1_A \otimes 1_B$.

Now let X and Y be topological spaces, and take the product: we have a diagram

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

where π_X and π_Y are projections. As $H^k(-;R)$ is functorial, this gives induced maps

$$\pi_X^* : H^k(X \times Y; R) \to H^k(X; R)$$
$$\pi_Y^* : H^k(X \times Y; R) \to H^k(Y; R)$$

for every k.

By using this, we can define a so-called cross product.

Definition 76.8.2. Let R be a ring, and X and Y spaces. Let π_X and π_Y be the projections of $X \times Y$ onto X and Y. Then the **cross product** is the map

$$H^\bullet(X; R) \otimes_R H^\bullet(Y; R) \xrightarrow{\times} H^\bullet(X \times Y; R)$$

acting on cocycles as follows: $\phi \times \psi = \pi_X^*(\phi) \smile \pi_Y^*(\psi)$.

This is just the most natural way to take a k-cocycle on X and an ℓ-cocycle on Y, and create a $(k+\ell)$-cocycle on the product space $X \times Y$.

Remark 76.8.3 — Of course, this definition coincides with the definition above using cellular cohomology, but the proof is omitted.

Theorem 76.8.4 (Künneth formula)

Let X and Y be CW complexes such that $H^k(Y; R)$ is a finitely generated free R-module for every k. Then the cross product is an isomorphism of anticommutative rings

$$H^\bullet(X; R) \otimes_R H^\bullet(Y; R) \to H^\bullet(X \times Y; R).$$

That is:

There is a one-to-one correspondence between pair of holes in X and Y and holes of $X \times Y$. Furthermore, the correspondence respects the cup product.

Where "holes" is to be understood as "generators of cohomology groups" in this case.

In any case, this finally lets us resolve the question set out at the beginning. We saw that $H_n(\mathbb{CP}^3) \cong H_n(S^2 \times S^4)$ for every n, and thus it follows that $H^n(\mathbb{CP}^3; \mathbb{Z}) \cong H^n(S^2 \times S^4; \mathbb{Z})$ too.

But now let us look at the cohomology rings. First, we have

$$H^\bullet(\mathbb{CP}^3; \mathbb{Z}) \cong \mathbb{Z}[\alpha]/(\alpha^4) \cong \mathbb{Z} \oplus \alpha\mathbb{Z} \oplus \alpha^2\mathbb{Z} \oplus \alpha^3\mathbb{Z}$$

where $|\alpha| = 2$; hence this is a graded ring generated by

- 1, in degree 0.
- α, in degree 2.

- α^2, in degree 4.
- α^3, in degree 6.

Now let's analyze

$$H^\bullet(S^2 \times S^4; \mathbb{Z}) \cong \mathbb{Z}[\beta]/(\beta^2) \otimes \mathbb{Z}[\gamma]/(\gamma^2).$$

It is thus generated thus by the following elements:

- $1 \otimes 1$, in degree 0.
- $\beta \otimes 1$, in degree 2.
- $1 \otimes \gamma$, in degree 4.
- $\beta \otimes \gamma$, in degree 6.

Again in each dimension we have the same abelian group. But notice that if we square $\beta \otimes 1$ we get

$$(\beta \otimes 1)(\beta \otimes 1) = \beta^2 \otimes 1 = 0.$$

Yet the degree 2 generator of $H^\bullet(\mathbb{CP}^3; \mathbb{Z})$ does not have this property. Hence these two graded rings are not isomorphic.

The nontrivial 4-cocycle $1 \otimes \gamma$ of $S^2 \times S^4$ is orthogonal to the 2-cocycle $\beta \otimes 1$, while the 4-cocycle α^2 of $\mathbb{CP}^3$ is the cup product $\alpha \smile \alpha$ of the 2-cocycle α with itself.

So it follows that $\mathbb{CP}^3$ and $S^2 \times S^4$ are not homotopy equivalent.

Exercise 76.8.5. Do the same procedure with $H^\bullet(\mathbb{RP}^3; \mathbb{Z}/2\mathbb{Z})$ and $H^\bullet(S^1 \times S^2; \mathbb{Z}/2\mathbb{Z})$. (Visualize $S^1 \times S^2$ as a thickened sphere with the outer and inner face fused together, and RP^3 as a closed 3-ball with opposing points on the boundary surface fused together. Try to stretch your mind and guess what the homology and cohomology groups are before formally compute it.)

§76.9 A few harder problems to think about

Problem 76A† (Symmetry of Betti numbers by Poincaré duality)**.** Let M be a smooth oriented compact n-manifold, and let b_k denote its Betti number. Prove that $b_k = b_{n-k}$.

Problem 76B. Show that $\mathbb{RP}^n$ is not orientable for even n.

Problem 76C. Show that $\mathbb{RP}^3$ is not homotopy equivalent to $\mathbb{RP}^2 \vee S^3$.

Problem 76D. Show that $S^m \vee S^n$ is not a deformation retract of $S^m \times S^n$ for any $m, n \geq 1$.