XVII

Category Theory

Part XVII: Contents

67 Objects and morphisms

I can't possibly hope to do category theory any justice in these few chapters; thus I'll just give a very high-level overview of how many of the concepts we've encountered so far can be re-cast into categorical terms. So I'll say what a category is, give some examples, then talk about a few things that categories can do. For my examples, I'll be drawing from all the previous chapters; feel free to skip over the examples corresponding to things you haven't seen.

If you're interested in category theory (like I was!), perhaps in what surprising results are true for general categories, I strongly recommend [Le14].

§67.1 Motivation: isomorphisms

From earlier chapters let's recall the definition of an *isomorphism* of two objects:

- Two groups G and H are isomorphic if there was a bijective homomorphism: equivalently, we wanted homomorphisms $\phi\colon G \to H$ and $\psi\colon H \to G$ which were mutual inverses, meaning $\phi \circ \psi = \mathrm{id}_H$ and $\psi \circ \phi = \mathrm{id}_G$.
- Two metric (or topological) spaces X and Y are isomorphic if there is a continuous bijection $f\colon X \to Y$ such that f^{-1} is also continuous.
- Two vector spaces V and W are isomorphic if there is a bijection $T\colon V \to W$ which is a linear map. Again, this can be re-cast as saying that T and T^{-1} are linear maps.
- Two rings R and S are isomorphic if there is a bijective ring homomorphism ϕ; again, we can re-cast this as two mutually inverse ring homomorphisms.

In each case we have some collections of objects and some maps, and the isomorphisms can be viewed as just maps. Let's use this to motivate the definition of a general *category*.

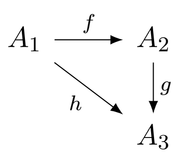
§67.2 Categories, and examples thereof

Prototypical example for this section: Grp *is possibly the most natural example.*

Definition 67.2.1. A **category** $\mathcal{A}$ consists of:

- A class of **objects**, denoted $\mathrm{obj}(\mathcal{A})$.
- For any two objects $A_1, A_2 \in \mathrm{obj}(\mathcal{A})$, a class of **arrows** (also called **morphisms** or **maps**) between them. We'll denote the set of these arrows by $\mathrm{Hom}_{\mathcal{A}}(A_1, A_2)$.
- For any $A_1, A_2, A_3 \in \mathrm{obj}(\mathcal{A})$, if $f\colon A_1 \to A_2$ is an arrow and $g\colon A_2 \to A_3$ is an arrow, we can compose these arrows to get an arrow $g \circ f\colon A_1 \to A_3$.

  We can represent this in a **commutative diagram**



where $h = g \circ f$. The composition operation $\circ$ is part of the data of $\mathcal{A}$; it must be associative. In the diagram above we say that h **factors** through A_2.

- Finally, every object $A \in \text{obj}(\mathcal{A})$ has a special **identity arrow** id_A; you can guess what it does.[1]

Abuse of Notation 67.2.2. From now on, by $A \in \mathcal{A}$ we'll mean $A \in \text{obj}(\mathcal{A})$.

Abuse of Notation 67.2.3. You can think of "class" as just "set". The reason we can't use the word "set" is because of some paradoxical issues with collections which are too large; Cantor's Paradox says there is no set of all sets. So referring to these by "class" is a way of sidestepping these issues.

Now and forever I'll be sloppy and assume all my categories are **locally small**, meaning that $\text{Hom}_{\mathcal{A}}(A_1, A_2)$ is a set for any $A_1, A_2 \in \mathcal{A}$. So elements of $\mathcal{A}$ may not form a set, but the set of morphisms between two *given* objects will always assumed to be a set.

Let's formalize the motivation we began with.

Example 67.2.4 (Basic examples of categories)

(a) There is a category of groups Grp. The data is
   - The objects of Grp are the groups.
   - The arrows of Grp are the homomorphisms between these groups.
   - The composition $\circ$ in Grp is function composition.

(b) In the same way we can conceive a category CRing of (commutative) rings.

(c) Similarly, there is a category Top of topological spaces, whose arrows are the continuous maps.

(d) There is a category Top_* of topological spaces with a *distinguished basepoint*; that is, a pair (X, x_0) where $x_0 \in X$. Arrows are continuous maps $f: X \to Y$ with $f(x_0) = y_0$.

(e) Similarly, there is a category Vect_k of vector spaces (possibly infinite-dimensional) over a field k, whose arrows are the linear maps. There is even a category FDVect_k of *finite-dimensional* vector spaces.

(f) We have a category Set of sets, where the arrows are *any* maps.

And of course, we can now define what an isomorphism is!

Definition 67.2.5. An arrow $A_1 \xrightarrow{f} A_2$ is an **isomorphism** if there exists $A_2 \xrightarrow{g} A_1$ such that $f \circ g = \text{id}_{A_2}$ and $g \circ f = \text{id}_{A_1}$. In that case we say A_1 and A_2 are **isomorphic**, hence $A_1 \cong A_2$.

Remark 67.2.6 — Note that in Set, $X \cong Y \iff |X| = |Y|$.

[1] To be painfully explicit: if $f: A' \to A$ is an arrow then $\text{id}_A \circ f = f$; similarly, if $g: A \to A'$ is an arrow then $g \circ \text{id}_A = g$.

> **Question 67.2.7.** Check that every object in a category is isomorphic to itself. (This is offensively easy.)

More importantly, this definition should strike you as a little impressive. We're able to define whether two groups (rings, spaces, etc.) are isomorphic solely by the functions between the objects. Indeed, one of the key themes in category theory (and even algebra) is that

One can learn about objects by the functions between them. Category theory takes this to the extreme by *only* looking at arrows, and ignoring what the objects themselves are.

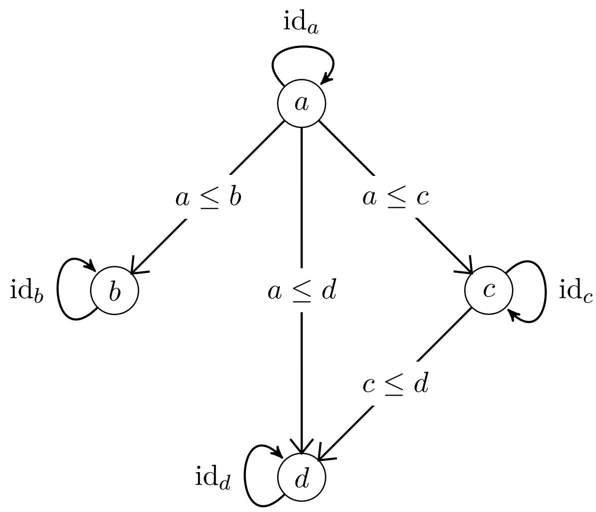
But there are some trickier interesting examples of categories.

Example 67.2.8 (Posets are categories)
Let $\mathcal{P}$ be a partially ordered set. We can construct a category P for it as follows:

- The objects of P are going to be the elements of $\mathcal{P}$.
- The arrows of P are defined as follows:
  - For every object $p \in P$, we add an identity arrow id_p, and
  - For any pair of distinct objects $p \le q$, we add a single arrow $p \to q$.

  There are no other arrows.
- There's only one way to do the composition. What is it?

For example, for the poset $\mathcal{P}$ on four objects $\{a, b, c, d\}$ with $a \le b$ and $a \le c \le d$, we get:



This illustrates the point that

The arrows of a category can be totally different from functions.

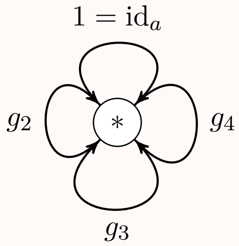
In fact, in a way that can be made precise, the term "concrete category" refers to one where the arrows really are "structure-preserving maps between sets", like Grp, Top, or CRing.

Question 67.2.9. Check that no two distinct objects of a poset are isomorphic.

Here's a second quite important example of a non-concrete category.

Example 67.2.10 (Important: groups are one-Object categories)

A group G can be interpreted as a category $\mathcal{G}$ with one object $*$, all of whose arrows are isomorphisms.



As [**Le14**] says:

> The first time you meet the idea that a group is a kind of category, it's tempting to dismiss it as a coincidence or a trick. It's not: there's real content. To see this, suppose your education had been shuffled and you took a course on category theory before ever learning what a group was. Someone comes to you and says:
>
> "There are these structures called 'groups', and the idea is this: a group is what you get when you collect together all the symmetries of a given thing."
>
> "What do you mean by a 'symmetry'?" you ask.
>
> "Well, a symmetry of an object X is a way of transforming X or mapping X into itself, in an invertible way."
>
> "Oh," you reply, "that's a special case of an idea I've met before. A category is the structure formed by *lots* of objects and mappings between them – not necessarily invertible. A group's just the very special case where you've only got one object, and all the maps happen to be invertible."

Exercise 67.2.11. Verify the above! That is, show that the data of a one-object category with all isomorphisms is the same as the data of a group.

Finally, here are some examples of categories you can make from other categories.

Example 67.2.12 (Deriving categories)

(a) Given a category $\mathcal{A}$, we can construct the **opposite category** $\mathcal{A}^{\mathrm{op}}$, which is the same as $\mathcal{A}$ but with all arrows reversed.

(b) Given categories $\mathcal{A}$ and $\mathcal{B}$, we can construct the **product category** $\mathcal{A} \times \mathcal{B}$ as follows: the objects are pairs (A, B) for $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and the arrows from

(A_1, B_1) to (A_2, B_2) are pairs

$$\left(A_1 \xrightarrow{f} A_2, B_1 \xrightarrow{g} B_2\right).$$

What do you think the composition and identities are?

§67.3 Special objects in categories

Prototypical example for this section: Set *has initial object* $\varnothing$ *and final object* $\{*\}$*. An element of* S *corresponds to a map* $\{*\} \to S$*.*

Certain objects in categories have special properties. Here are a couple examples.

Example 67.3.1 (Initial object)

An **initial object** of $\mathcal{A}$ is an object $A_{\text{init}} \in \mathcal{A}$ such that for any $A \in \mathcal{A}$ (possibly $A = A_{\text{init}}$), there is exactly one arrow from A_{init} to A. For example,

(a) The initial object of Set is the empty set $\varnothing$.

(b) The initial object of Grp is the trivial group $\{1\}$.

(c) The initial object of CRing is the ring $\mathbb{Z}$ (recall that ring homomorphisms $R \to S$ map 1_R to 1_S).

(d) The initial object of Top is the empty space.

(e) The initial object of a partially ordered set is its smallest element, if one exists.

We will usually refer to "the" initial object of a category, since:

Exercise 67.3.2 (Important!)**.** Show that any two initial objects A_1, A_2 of $\mathcal{A}$ are *uniquely isomorphic* meaning there is a unique isomorphism between them.

Remark 67.3.3 — In mathematics, we usually neither know nor care if two objects are actually equal or whether they are isomorphic. For example, there are many competing ways to define $\mathbb{R}$, but we still just refer to it as "the" real numbers.

Thus when we define categorical notions, we would like to check they are unique up to isomorphism. This is really clean in the language of categories, and definitions often cause objects to be unique up to isomorphism for elegant reasons like the above.

One can take the "dual" notion, a terminal object.

Example 67.3.4 (Terminal object)

A **terminal object** of $\mathcal{A}$ is an object $A_{\text{final}} \in \mathcal{A}$ such that for any $A \in \mathcal{A}$ (possibly $A = A_{\text{final}}$), there is exactly one arrow from A to A_{final}. For example,

(a) The terminal object of Set is the singleton set $\{*\}$. (There are many singleton sets, of course, but *as sets* they are all isomorphic!)

(b) The terminal object of Grp is the trivial group $\{1\}$.

(c) The terminal object of CRing is the zero ring 0. (Recall that ring homomorphisms $R \to S$ must map 1_R to 1_S).

(d) The terminal object of Top is the single-point space.

(e) The terminal object of a partially ordered set is its maximal element, if one exists.

Again, terminal objects are unique up to isomorphism. The reader is invited to repeat the proof from the preceding exercise here. However, we can illustrate more strongly the notion of duality to give a short proof.

Question 67.3.5. Verify that terminal objects of $\mathcal{A}$ are equivalent to initial objects of $\mathcal{A}^{\text{op}}$. Thus terminal objects of $\mathcal{A}$ are unique up to isomorphism.

In general, one can consider in this way the dual of *any* categorical notion: properties of $\mathcal{A}$ can all be translated to dual properties of $\mathcal{A}^{\text{op}}$ (often by adding the prefix "co" in front).

One last neat construction: suppose we're working in a concrete category, meaning (loosely) that the objects are "sets with additional structure". Now suppose you're sick of maps and just want to think about elements of these sets. Well, I won't let you do that since you're reading a category theory chapter, but I will offer you some advice:

- In Set, arrows from $\{*\}$ to S correspond to elements of S.
- In Top, arrows from $\{*\}$ to X correspond to points of X.
- In Grp, arrows from $\mathbb{Z}$ to G correspond to elements of G.
- In CRing, arrows from $\mathbb{Z}[x]$ to R correspond to elements of R.

and so on. So in most concrete categories, you can think of elements as functions from special sets to the set in question. In each of these cases we call the object in question a **free object**.

§67.4 Binary products

Prototypical example for this section: $X \times Y$ in most concrete categories is the set-theoretic product.

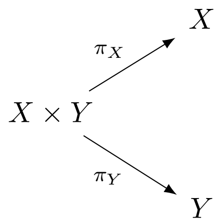
The "universal property" is a way of describing objects in terms of maps in such a way that it defines the object up to unique isomorphism (much the same as the initial and terminal objects).

To show how this works in general, let me give a concrete example. Suppose I'm in a category – let's say Set for now. I have two sets X and Y, and I want to construct the Cartesian product $X \times Y$ as we know it. The philosophy of category theory dictates that I should talk about maps only, and avoid referring to anything about the sets themselves. How might I do this?

Well, let's think about maps into $X \times Y$. The key observation is that

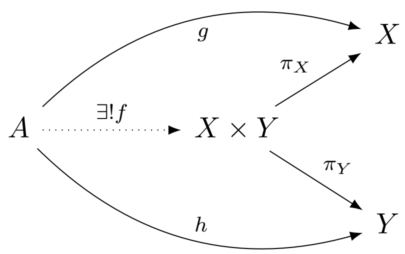
A function $A \xrightarrow{f} X \times Y$ amounts to a pair of functions $(A \xrightarrow{g} X, A \xrightarrow{h} Y)$.

Put another way, there is a natural projection map $X \times Y \to X$ and $X \times Y \to Y$:



(We have to do this in terms of projection maps rather than elements, because category theory forces us to talk about arrows.) Now how do I add A to this diagram? The point is that there is a bijection between functions $A \xrightarrow{f} X \times Y$ and pairs (g, h) of functions. Thus for every pair $A \xrightarrow{g} X$ and $A \xrightarrow{h} Y$ there is a *unique* function $A \xrightarrow{f} X \times Y$.

But $X \times Y$ is special in that it is "universal": for any *other* set A, if you give me functions $A \to X$ and $A \to Y$, I can use it build a *unique* function $A \to X \times Y$. Picture:



We can do this in any general category, defining a so-called product.

Definition 67.4.1. Let X and Y be objects in any category $\mathcal{A}$. The **product** consists of an object $X \times Y$ and arrows π_X, π_Y to X and Y (thought of as projection). We require that for any object A and arrows $A \xrightarrow{g} X$, $A \xrightarrow{h} Y$, there is a *unique* function $A \xrightarrow{f} X \times Y$ such that the above diagram commutes.

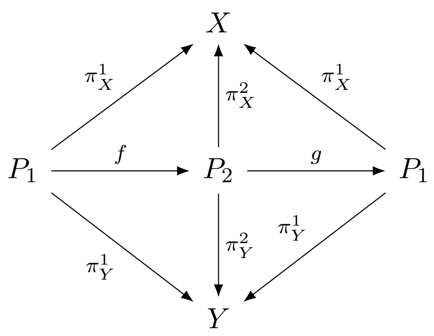
Abuse of Notation 67.4.2. Strictly speaking, the product should consist of *both* the object $X \times Y$ and the projection maps π_X and π_Y. However, if π_X and π_Y are understood, then we often use $X \times Y$ to refer to the object, and refer to it also as the product.

Products do not always exist; for example, take a category with just two objects and no non-identity morphisms. Nonetheless:

Proposition 67.4.3 (Uniqueness of products)

When they exist, products are unique up to isomorphism: given two products P_1 and P_2 of X and Y there is an isomorphism between the two objects.

Proof. This is very similar to the proof that initial objects are unique up to unique isomorphism. Consider two such objects P_1 and P_2, and the associated projection maps. So, we have a diagram



There are unique morphisms f and g between P_1 and P_2 that make the entire diagram commute, according to the universal property.

On the other hand, look at $g \circ f$ and focus on just the outer square. Observe that $g \circ f$ is a map which makes the outer square commute, so by the universal property of P_1 it is the only one. But id_{P_1} works as well. Thus $\mathrm{id}_{P_1} = g \circ f$. Similarly, $f \circ g = \mathrm{id}_{P_2}$ so f and g are isomorphisms. □

Abuse of Notation 67.4.4. Actually, this is not really the morally correct theorem; since we've only showed the objects P_1 and P_2 are isomorphic and have not made any assertion about the projection maps. But I haven't (and won't) define isomorphism of the entire product, and so in what follows if I say "P_1 and P_2 are isomorphic" I really just mean the objects are isomorphic.

> **Exercise 67.4.5.** In fact, show the products are unique up to *unique* isomorphism: the f and g above are the only isomorphisms between the objects P_1 and P_2 respecting the projections.

The nice fact about this "universal property" mindset is that we don't have to give explicit constructions; assuming existence, the "universal property" allows us to bypass all this work by saying "the object with these properties is unique up to unique isomorphism", thus we don't need to understand the internal workings of the object to use its properties.

Of course, that's not to say we can't give concrete examples.

Example 67.4.6 (Examples of products)

(a) In Set, the product of two sets X and Y is their Cartesian product $X \times Y$.

(b) In Grp, the product of G, H is the group product $G \times H$.

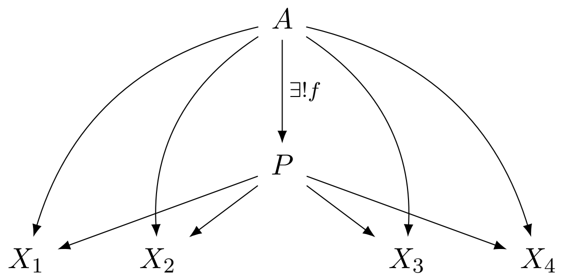
(c) In Vect_k, the product of V and W is $V \oplus W$.

(d) In CRing, the product of R and S is appropriately the ring product $R \times S$.

(e) Let $\mathcal{P}$ be a poset interpreted as a category. Then the product of two objects x and y is the **greatest lower bound**; for example,
- If the poset is $(\mathbb{R}, \le)$ then it's $\min\{x, y\}$.
- If the poset is the subsets of a finite set by inclusion, then it's $x \cap y$.
- If the poset is the positive integers ordered by division, then it's $\gcd(x, y)$.

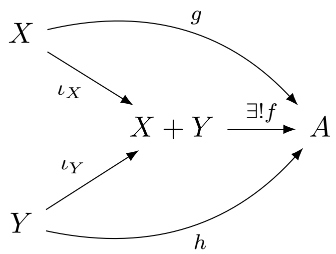
Of course, we can define products of more than just one object. Consider a set of objects $(X_i)_{i \in I}$ in a category $\mathcal{A}$. We define a **cone** on the X_i to be an object A with some "projection" maps to each X_i. Then the **product** is a cone P which is "universal" in the same sense as before: given any other cone A there is a unique map $A \to P$ making the diagram commute. In short, a product is a "universal cone".

The picture of this is



See also Problem 67C.

One can also do the dual construction to get a **coproduct**: given X and Y, it's the object $X + Y$ together with maps $X \xrightarrow{\iota_X} X + Y$ and $Y \xrightarrow{\iota_Y} X + Y$ (that's Greek iota, think inclusion) such that for any object A and maps $X \xrightarrow{g} A$, $Y \xrightarrow{h} A$ there is a unique f for which



commutes. We'll leave some of the concrete examples as an exercise this time, for example:

> **Exercise 67.4.7.** Describe the coproduct in Set.

Predictable terminology: a coproduct is a universal **cocone**.

Spoiler alert later on: this construction can be generalized vastly to so-called "limits", and we'll do so later on.

§67.5 Monic and epic maps

The notion of "injective" doesn't make sense in an arbitrary category since arrows need not be functions. The correct categorical notion is:

Definition 67.5.1. A map $X \xrightarrow{f} Y$ is **monic** (or a monomorphism) if for any commutative diagram

$$A \underset{h}{\overset{g}{\rightrightarrows}} X \xrightarrow{f} Y$$

we must have $g = h$. In other words, $f \circ g = f \circ h \implies g = h$.

> **Question 67.5.2.** Verify that in a *concrete* category, injective $\implies$ monic.

Question 67.5.3. Show that the composition of two monic maps is monic.

In most but not all situations, the converse is also true.

Exercise 67.5.4. Show that in Set, Grp, CRing, monic implies injective. (Take $A = \{*\}$, $A = \mathbb{Z}$, $A = \mathbb{Z}[x]$.)

More generally, as we said before there are many categories with a "free" object that you can use to think of as elements. An element of a set is a function $1 \to S$, and element of a ring is a function $\mathbb{Z}[x] \to R$, et cetera. In all these categories, the definition of monic literally reads "f is injective on $\operatorname{Hom}_{\mathcal{A}}(A, X)$". So in these categories, "monic" and "injective" coincide.

That said, here is the standard counterexample. An additive abelian group $G = (G, +)$ is called *divisible* if for every $x \in G$ and integer $n > 0$ there exists $y \in G$ with $ny = x$. Let DivAbGrp be the category of such groups.

Exercise 67.5.5. Show that the projection $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is monic but not injective.

Of course, we can also take the dual notion.

Definition 67.5.6. A map $X \xrightarrow{f} Y$ is **epic** (or an epimorphism) if for any commutative diagram

$$X \xrightarrow{f} Y \underset{h}{\overset{g}{\rightrightarrows}} A$$

we must have $g = h$. In other words, $g \circ f = h \circ f \implies g = h$.

This is kind of like surjectivity, although it's a little farther than last time. Note that in concrete categories, surjective $\implies$ epic.

Exercise 67.5.7. Show that in Set, Grp, Ab, Vect_k, Top, the notions of epic and surjective coincide. (For Set, take $A = \{0, 1\}$.)

However, there are more cases where it fails. Most notably:

Example 67.5.8 (Epic but not surjective)

(a) In CRing, for instance, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is epic (and not surjective). Indeed, if two homomorphisms $\mathbb{Q} \to A$ agree on every integer then they agree everywhere (why?),

(b) In the category of *Hausdorff* topological spaces (every two points have disjoint open neighborhoods), in fact epic $\iff$ dense image (like $\mathbb{Q} \hookrightarrow \mathbb{R}$).

Thus failures arise when a function $f \colon X \to Y$ can be determined by just some of the points of X.

§67.6 A few harder problems to think about

Problem 67A. In the category Vect_k of k-vector spaces (for a field k), what are the initial and terminal objects?

Problem 67B†. What is the coproduct $X + Y$ in the categories Set, Vect_k, and a poset?

Problem 67C. In any category $\mathcal{A}$ where all products exist, show that

$$(X \times Y) \times Z \cong X \times (Y \times Z)$$

where X, Y, Z are arbitrary objects. (Here both sides refer to the objects, as in Abuse of Notation 67.4.2.)

Problem 67D. Consider a category $\mathcal{A}$ with a **zero object**, meaning an object which is both initial and terminal. Given objects X and Y in A, prove that the projection $X \times Y \to X$ is epic.

68 Functors and natural transformations

Functors are maps between categories; natural transformations are maps between functors.

§68.1 Many examples of functors

Prototypical example for this section: Forgetful functors; fundamental groups; $\bullet^\vee$.

Here's the point of a functor:

Pretty much any time you make an object out of another object, you get a functor.

Before I give you a formal definition, let me list (informally) some examples. (You'll notice some of them have opposite categories $\mathcal{A}^{\text{op}}$ appearing in places. Don't worry about those for now; you'll see why in a moment.)

- Given a group G (or vector space, field, ...), we can take its underlying set S; this is a functor from $\mathsf{Grp} \to \mathsf{Set}$.
- Given a set S we can consider a vector space with basis S; this is a functor from $\mathsf{Set} \to \mathsf{Vect}$.
- Given a vector space V we can consider its dual space $V^\vee$. This is a functor $\mathsf{Vect}_k^{\text{op}} \to \mathsf{Vect}_k$.
- Tensor products give a functor from $\mathsf{Vect}_k \times \mathsf{Vect}_k \to \mathsf{Vect}_k$.
- Given a set S, we can build its power set, giving a functor $\mathsf{Set} \to \mathsf{Set}$.
- In algebraic topology, we take a topological space X and build several groups $H_1(X)$, $\pi_1(X)$, etc. associated to it. All these group constructions are functors $\mathsf{Top} \to \mathsf{Grp}$.
- Sets of homomorphisms: let $\mathcal{A}$ be a category.
  - Given two vector spaces V_1 and V_2 over k, we construct the abelian group of linear maps $V_1 \to V_2$. This is a functor from $\mathsf{Vect}_k^{\text{op}} \times \mathsf{Vect}_k \to \mathsf{AbGrp}$.
  - More generally for any category $\mathcal{A}$ we can take pairs (A_1, A_2) of objects and obtain a set $\operatorname{Hom}_{\mathcal{A}}(A_1, A_2)$. This turns out to be a functor $\mathcal{A}^{\text{op}} \times \mathcal{A} \to \mathsf{Set}$.
  - The above operation has two "slots". If we "pre-fill" the first slots, then we get a functor $\mathcal{A} \to \mathsf{Set}$. That is, by fixing $A \in \mathcal{A}$, we obtain a functor (called H^A) from $\mathcal{A} \to \mathsf{Set}$ by sending $A' \in \mathcal{A}$ to $\operatorname{Hom}_{\mathcal{A}}(A, A')$. This is called the covariant Yoneda functor (explained later).
  - As we saw above, for every $A \in \mathcal{A}$ we obtain a functor $H^A \colon \mathcal{A} \to \mathsf{Set}$. It turns out we can construct a category $[\mathcal{A}, \mathsf{Set}]$ whose elements are functors $\mathcal{A} \to \mathsf{Set}$; in that case, we now have a functor $\mathcal{A}^{\text{op}} \to [\mathcal{A}, \mathsf{Set}]$.

That having said, here are some non-functors. Just so that when you see a theorem that says "F is a functor" (in other words, "F is functorial"), you should read it as "F has a deep hidden symmetry behind it! This is very nice!" instead of "this theorem is trivial".

What is that deep symmetry? Keep reading.

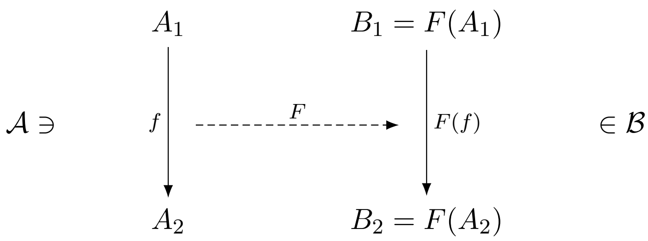
- Given a group G, we can build its automorphism group $\operatorname{Aut}(G)$. But this is not a functor in any natural way.
- Given a group G, we can build its center $Z(G)$, which is the set of elements in G that commutes with everything in G. Again, this is not a functor in any natural way.[1]
- The operation of taking the dual space above is a contravariant functor $\mathsf{Vect}_k^{\mathrm{op}} \to \mathsf{Vect}_k$, but it isn't a covariant functor $\mathsf{Vect}_k \to \mathsf{Vect}_k$. (Don't worry what a contravariant functor is for now.)

§68.2 Covariant functors

Prototypical example for this section: Forgetful/free functors, ...

Category theorists are always asking "what are the maps?", and so we can now think about maps between categories.

Definition 68.2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be categories. Of course, a **functor** F takes every object of $\mathcal{A}$ to an object of $\mathcal{B}$. In addition, though, it must take every arrow $A_1 \xrightarrow{f} A_2$ to an arrow $F(A_1) \xrightarrow{F(f)} F(A_2)$. You can picture this as follows.



(I'll try to use dotted arrows for functors, which cross different categories, for emphasis.) It needs to satisfy the "naturality" requirements:

- Identity arrows get sent to identity arrows: for each identity arrow id_A, we have $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.
- The functor respects composition: if $A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3$ are arrows in $\mathcal{A}$, then $F(g \circ f) = F(g) \circ F(f)$.

So the idea is:

Whenever we naturally make an object $A \in \mathcal{A}$ into an object of $B \in \mathcal{B}$, there should usually be a natural way to transform a map $A_1 \to A_2$ into a map $B_1 \to B_2$.

Let's see some examples of this.

[1]It is easy to find a counterexample based on properties of functor — in particular, identity maps get sent to identity maps. See https://math.stackexchange.com/q/158438 for a proof.

Example 68.2.2 (Free and forgetful functors)
Note that these are both informal terms, and don't have a rigid definition.

(a) We talked about a **forgetful functor** earlier, which takes the underlying set of a category like Vect_k. Let's call it $U\colon \mathsf{Vect}_k \to \mathsf{Set}$.

Now, given a map $T\colon V_1 \to V_2$ in Vect_k, there is an obvious $U(T)\colon U(V_1) \to U(V_2)$ which is just the set-theoretic map corresponding to T.

Similarly there are forgetful functors from Grp, CRing, etc., to Set. There is even a forgetful functor $\mathsf{CRing} \to \mathsf{Grp}$: send a ring R to the abelian group $(R, +)$. The common theme is that we are "forgetting" structure from the original category.

(b) We also talked about a **free functor** in the example. A free functor $F\colon \mathsf{Set} \to \mathsf{Vect}_k$ can be taken by considering $F(S)$ to be the vector space with basis S. Now, given a map $f\colon S \to T$, what is the obvious map $F(S) \to F(T)$? Simple: take each basis element $s \in S$ to the basis element $f(s) \in T$.

Similarly, we can define $F\colon \mathsf{Set} \to \mathsf{Grp}$ by taking the free group generated by a set S.

Remark 68.2.3 — There is also a notion of "injective" and "surjective" for functors (on arrows) as follows. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is **faithful** (resp. **full**) if for any A_1, A_2, $F\colon \operatorname{Hom}_{\mathcal{A}}(A_1, A_2) \to \operatorname{Hom}_{\mathcal{B}}(FA_1, FA_2)$ is injective (resp. surjective).[a]

We can use this to give an exact definition of concrete category: it's a category with a faithful (forgetful) functor $U\colon \mathcal{A} \to \mathsf{Set}$.

[a] Again, experts might object that $\operatorname{Hom}_{\mathcal{A}}(A_1, A_2)$ or $\operatorname{Hom}_{\mathcal{B}}(FA_1, FA_2)$ may be proper classes instead of sets, but I am assuming everything is locally small.

Example 68.2.4 (Functors from $\mathcal{G}$)
Let G be a group and $\mathcal{G} = \{*\}$ be the associated one-object category.

(a) Consider a functor $F\colon \mathcal{G} \to \mathsf{Set}$, and let $S = F(*)$. Then the data of F corresponds to putting a *group action* of G on S.

(b) Consider a functor $F\colon \mathcal{G} \to \mathsf{FDVect}_k$, and let $V = F(*)$ have dimension n. Then the data of F corresponds to embedding G as a subgroup of the $n \times n$ matrices (i.e. the linear maps $V \to V$). This is one way groups historically arose; the theory of viewing groups as matrices forms the field of representation theory.

(c) Let H be a group and construct $\mathcal{H}$ the same way. Then functors $\mathcal{G} \to \mathcal{H}$ correspond to homomorphisms $G \to H$.

Exercise 68.2.5. Check the above group-based functors work as advertised.

Here's a more involved example. If you find it confusing, skip it and come back after reading about its contravariant version.

Example 68.2.6 (Covariant Yoneda functor)

Fix an $A \in \mathcal{A}$. For a category $\mathcal{A}$, define the **covariant Yoneda functor** $H^A : \mathcal{A} \to \mathsf{Set}$ by defining

$$H^A(A_1) := \operatorname{Hom}_{\mathcal{A}}(A, A_1) \in \mathsf{Set}.$$

Hence each A_1 is sent to the *arrows from A to A_1*; so H^A **describes how A sees the world**.

Now we want to specify how H^A behaves on arrows. For each arrow $A_1 \xrightarrow{f} A_2$, we need to specify Set-map $\operatorname{Hom}_{\mathcal{A}}(A, A_1) \to \operatorname{Hom}(A, A_2)$; in other words, we need to send an arrow $A \xrightarrow{p} A_1$ to an arrow $A \to A_2$. There's only one reasonable way to do this: take the composition

$$A \xrightarrow{p} A_1 \xrightarrow{f} A_2.$$

In other words, $H_A(f)$ is $p \mapsto f \circ p$. In still other words, $H_A(f) = f \circ -$; the $-$ is a slot for the input to go into.

As another example:

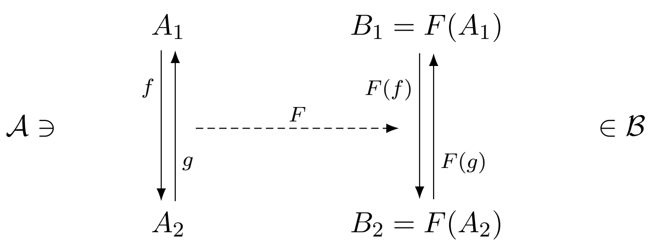
Question 68.2.7. If $\mathcal{P}$ and $\mathcal{Q}$ are posets interpreted as categories, what does a functor from $\mathcal{P}$ to $\mathcal{Q}$ represent?

Now, let me explain why we might care. Consider the following "obvious" fact: if G and H are isomorphic groups, then they have the same size. We can formalize it by saying: if $G \cong H$ in Grp and $U : \mathsf{Grp} \to \mathsf{Set}$ is the forgetful functor (mapping each group to its underlying set), then $U(G) \cong U(H)$. The beauty of category theory shows itself: this in fact works *for any functors and categories*, and the proof is done solely through arrows:

Theorem 68.2.8 (Functors preserve isomorphism)

If $A_1 \cong A_2$ are isomorphic objects in $\mathcal{A}$ and $F : \mathcal{A} \to \mathcal{B}$ is a functor then $F(A_1) \cong F(A_2)$.

Proof. Try it yourself! The picture is:



You'll need to use both key properties of functors: they preserve composition and the identity map. ◻

This will give us a great intuition in the future, because

(i) Almost every operation we do in our lifetime will be a functor, and

(ii) We now know that functors take isomorphic objects to isomorphic objects.

Thus, we now automatically know that basically any "reasonable" operation we do will preserve isomorphism (where "reasonable" means that it's a functor). This is super convenient in algebraic topology, for example; see Theorem 65.6.2, where we get for free that homotopic spaces have isomorphic fundamental groups.

Remark 68.2.9 — This lets us construct a category Cat whose objects are categories and arrows are functors.

§68.3 Covariant functors as indexed family of objects

Instead of viewing functor as a *function*, sometimes it is more convenient to view a functor as an *object* (or a family of objects).

For sets A and B, sometimes the notation A^B is used to denote the set $\operatorname{Hom}(B, A)$ being the set of all functions from B to A. This notation is natural because, for finite sets A and B, then $|\operatorname{Hom}(B, A)| = |A|^{|B|}$.

That said, the product set $A \times A$ is sometimes also denoted A^2. Is there a relation?

Certainly! We define the set $\mathbf{2} = \{0, 1\}$ (or any set of two elements). Then we have $|\mathbf{2}| = 2$. It is not difficult to see there is a correspondence between A^2 and $\operatorname{Hom}(\mathbf{2}, A)$.

Now, let $\mathcal{A}$ be a category. Define the category $\mathcal{A} \times \mathcal{A} = \mathcal{A}^2$ the obvious way:

- The objects of $\mathcal{A}^2$ are pairs of objects (A_1, A_2) with $A_1, A_2 \in \mathcal{A}$,
- The morphisms are pairs of morphisms...

Exercise 68.3.1. For $X, Y \in \mathsf{Top}$, we can define the product space $X \times Y \in \mathsf{Top}$. This gives a functor $\mathsf{Top}^2 \to \mathsf{Top}$. Verify this. (From a pair of maps $(f, g) : (X_1, Y_1) \to (X_2, Y_2)$ in Top^2, how do we get a map $X_1 \times Y_1 \to X_2 \times Y_2$? Check this map is continuous i.e. it is indeed a morphism in Top.)

Similar to above, each object in $\mathcal{A}^2$ should correspond to some sort of function $f : \mathsf{2} \to \mathcal{A}$. But a function's codomain must be an object... $\mathcal{A}$ is a category, so f should be a functor!

So we can make a category $\mathsf{2}$, and we have $F : \mathsf{2} \to \mathcal{A}$. There is only one reasonable way to define $\mathsf{2}$ that do what we want:[2]

- The objects are $\{0, 1\}$;
- There is no morphism, except id_0 and id_1.

More generally,

A functor $F : \mathcal{A} \to \mathcal{B}$ can be viewed as an indexed collection of objects $\{B_A \in \mathcal{B}\}_{A \in \mathcal{A}}$.

This can be most easily seen for a presheaf: "a contravariant functor $\operatorname{OpenSets}(X)^{\mathrm{op}} \to \mathsf{Rings}$" means "a family of rings indexed by open sets of X, satisfying certain niceness conditions".

In fact, just as $\mathcal{A}^2$ is a category, the functors $\mathsf{2} \to \mathcal{A}$ also forms a category. We will see this in Section 68.7.

[2] This is **different** from the category $\mathbf{2}$ that we will define later for natural transformation! Be careful.

§68.4 Contravariant functors

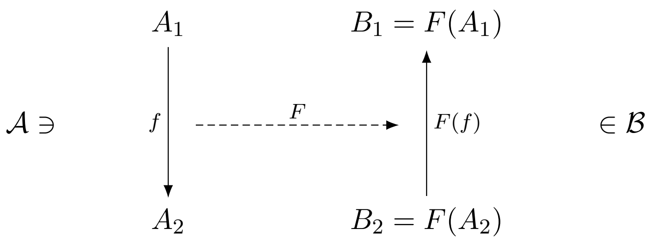
Prototypical example for this section: Dual spaces, contravariant Yoneda functor, etc.

Now I have to explain what the opposite categories were doing earlier. In all the previous examples, we took an arrow $A_1 \to A_2$, and it became an arrow $F(A_1) \to F(A_2)$. Sometimes, however, the arrow in fact goes the other way: we get an arrow $F(A_2) \to F(A_1)$ instead. In other words, instead of just getting a functor $\mathcal{A} \to \mathcal{B}$ we ended up with a functor $\mathcal{A}^{\text{op}} \to \mathcal{B}$.

These functors have a name:

Definition 68.4.1. A **contravariant functor** from $\mathcal{A}$ to $\mathcal{B}$ is a functor $F\colon \mathcal{A}^{\text{op}} \to \mathcal{B}$. (Note that we do *not* write "contravariant functor $F\colon \mathcal{A} \to \mathcal{B}$", since that would be confusing; the function notation will always use the correct domain and codomain.)

Pictorially:



For emphasis, a usual functor is often called a **covariant functor**. (The word "functor" with no adjective always refers to covariant.)

Let's see why this might happen.

Example 68.4.2 ($V \mapsto V^\vee$ is contravariant)

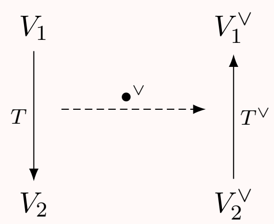
Consider the functor $\mathsf{Vect}_k \to \mathsf{Vect}_k$ by $V \mapsto V^\vee$.

If we were trying to specify a covariant functor, we would need, for every linear map $T\colon V_1 \to V_2$, a linear map $T^\vee\colon V_1^\vee \to V_2^\vee$. But recall that $V_1^\vee = \operatorname{Hom}(V_1, k)$ and $V_2^\vee = \operatorname{Hom}(V_2, k)$: there's no easy way to get an obvious map from left to right.

However, there *is* an obvious map from right to left: given $\xi_2\colon V_2 \to k$, we can easily give a map from $V_1 \to k$: just compose with T! In other words, there is a very natural map $V_2^\vee \to V_1^\vee$ according to the composition

$$V_1 \xrightarrow{T} V_2 \xrightarrow{\xi_2} k$$

In summary, a map $T\colon V_1 \to V_2$ induces naturally a map $T^\vee\colon V_2^\vee \to V_1^\vee$ in the opposite direction. So the contravariant functor looks like:



We can generalize the example above in any category by replacing the field k with any chosen object $A \in \mathcal{A}$.

Example 68.4.3 (Contravariant Yoneda functor)

The **contravariant Yoneda functor** on $\mathcal{A}$, denoted $H_A \colon \mathcal{A}^{\text{op}} \to \mathsf{Set}$, is used to describe how objects of $\mathcal{A}$ see A. For each $X \in \mathcal{A}$ it puts

$$H_A(X) := \operatorname{Hom}_{\mathcal{A}}(X, A) \in \mathsf{Set}.$$

For $X \xrightarrow{f} Y$ in $\mathcal{A}$, the map $H_A(f)$ sends each arrow $Y \xrightarrow{p} A \in \operatorname{Hom}_{\mathcal{A}}(Y, A)$ to

$$X \xrightarrow{f} Y \xrightarrow{p} A \quad \in \operatorname{Hom}_{\mathcal{A}}(X, A)$$

as we did above. Thus $H_A(f)$ is an arrow from $\operatorname{Hom}_{\mathcal{A}}(Y, A) \to \operatorname{Hom}_{\mathcal{A}}(X, A)$. (Note the flipping!)

Exercise 68.4.4. Check now the claim that $\mathcal{A}^{\text{op}} \times \mathcal{A} \to \mathsf{Set}$ by $(A_1, A_2) \mapsto \operatorname{Hom}(A_1, A_2)$ is in fact a functor.

§68.5 Equivalence of categories

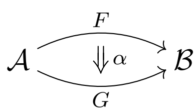
fully faithful and essentially surjective

§68.6 (Optional) Natural transformations

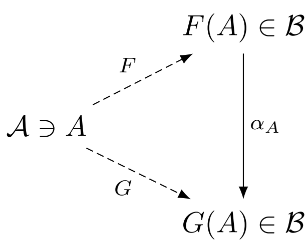
We made categories to keep track of objects and maps, then went a little crazy and asked "what are the maps between categories?" to get functors. Now we'll ask "what are the maps between functors?" to get natural transformations.

It might sound terrifying that we're drawing arrows between functors, but this is actually an old idea. Recall that given two paths $\alpha, \beta \colon [0,1] \to X$, we built a path-homotopy by "continuously deforming" the path α to β; this could be viewed as a function $[0,1] \times [0,1] \to X$. The definition of a natural transformation is similar: we want to pull F to G along a series of arrows in the target space $\mathcal{B}$.

Definition 68.6.1. Let $F, G \colon \mathcal{A} \to \mathcal{B}$ be two functors. A **natural transformation** α from F to G, denoted



consists of, for each $A \in \mathcal{A}$ an arrow $\alpha_A \in \operatorname{Hom}_{\mathcal{B}}(F(A), G(A))$, which is called the **component** of α at A. Pictorially, it looks like this:

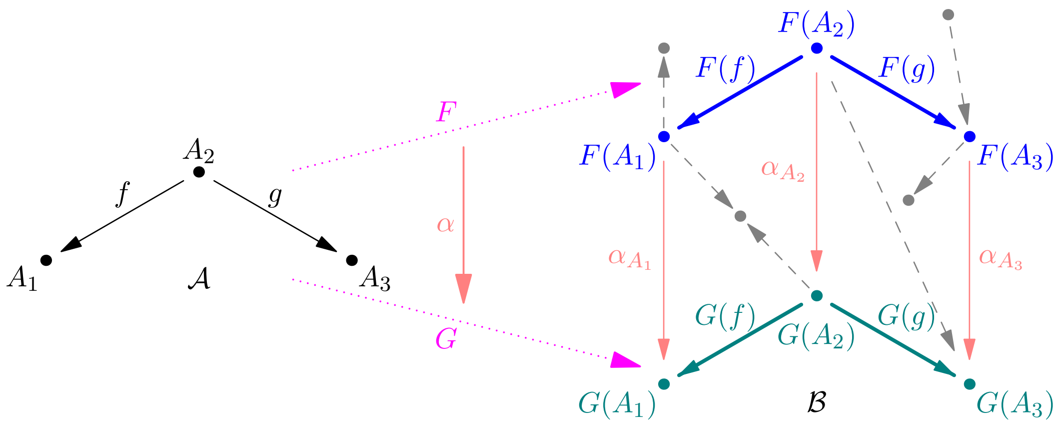


These α_A are subject to the "naturality" requirement that for any $A_1 \xrightarrow{f} A_2$, the diagram

$$\begin{array}{ccc} F(A_1) & \xrightarrow{F(f)} & F(A_2) \\ \downarrow{\scriptstyle \alpha_{A_1}} & & \downarrow{\scriptstyle \alpha_{A_2}} \\ G(A_1) & \xrightarrow[G(f)]{} & G(A_2) \end{array}$$

commutes.

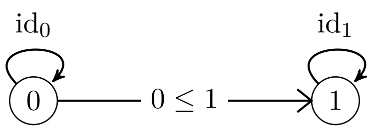
The arrow α_A represents the path that $F(A)$ takes to get to $G(A)$ (just as in a path-homotopy from α to β each *point* $\alpha(t)$ gets deformed to the *point* $\beta(t)$ continuously). A picture might help: consider



Here $\mathcal{A}$ is the small category with three elements and two non-identity arrows f, g (I've omitted the identity arrows for simplicity). The images of $\mathcal{A}$ under F and G are the blue and green "subcategories" of $\mathcal{B}$. Note that $\mathcal{B}$ could potentially have many more objects and arrows in it (grey). The natural transformation α (red) selects an arrow of $\mathcal{B}$ from each $F(A)$ to the corresponding $G(A)$, dragging the entire image of F to the image of G. Finally, we require that any diagram formed by the blue, red, and green arrows is commutative (naturality), so the natural transformation is really "natural".

There is a second equivalent definition that looks much more like the homotopy.

Definition 68.6.2. Let $\mathbf{2}$ denote the category generated by a poset with two elements $0 \le 1$, that is,



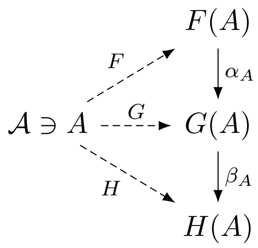
Then a *natural transformation* $\mathcal{A} \underset{G}{\overset{F}{\Downarrow \alpha}} \mathcal{B}$ is just a functor $\alpha \colon \mathcal{A} \times \mathbf{2} \to \mathcal{B}$ satisfying

$$\alpha(A, 0) = F(A), \quad \alpha(f, 0) = F(f) \quad \text{and} \quad \alpha(A, 1) = G(A), \quad \alpha(f, 1) = G(f).$$

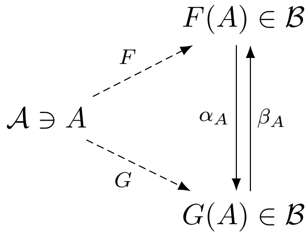
More succinctly, $\alpha(-, 0) = F$, $\alpha(-, 1) = G$.

The proof that these are equivalent is left as a practice problem.

Naturally, two natural transformations $\alpha \colon F \to G$ and $\beta \colon G \to H$ can get composed.

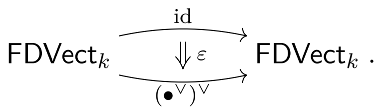


Now suppose α is a natural transformation such that α_A is an isomorphism for each A. In this way, we can construct an inverse arrow β_A to it.



In this case, we say α is a **natural isomorphism**. We can then say that $F(A) \cong G(A)$ **naturally** in A. (And β is an isomorphism too!) This means that the functors F and G are "really the same": not only are they isomorphic on the level of objects, but these isomorphisms are "natural". As a result of this, we also write $F \cong G$ to mean that the functors are naturally isomorphic.

This is what it really means when we say that "there is a natural / canonical isomorphism". For example, I claimed earlier (in Problem 15A*) that there was a canonical isomorphism $(V^\vee)^\vee \cong V$, and mumbled something about "not having to pick a basis" and "God-given". Category theory, amazingly, lets us formalize this: it just says that $(V^\vee)^\vee \cong \text{id}(V)$ naturally in $V \in \mathsf{FDVect}_k$. Really, we have a natural transformation



where the component ε_V is given by $v \mapsto \text{ev}_v$ (as discussed earlier, the fact that it is an isomorphism follows from the fact that V and $(V^\vee)^\vee$ have equal dimensions and ε_V is injective).

Another example can be found in Remark 71.2.8.

§68.7 (Optional) The Yoneda lemma

Now that I have natural transformations, I can define:

Definition 68.7.1. The **functor category** of two categories $\mathcal{A}$ and $\mathcal{B}$, denoted $[\mathcal{A}, \mathcal{B}]$, is defined as follows:

- The objects of $[\mathcal{A}, \mathcal{B}]$ are (covariant) functors $F\colon \mathcal{A} \to \mathcal{B}$, and
- The morphisms are natural transformations $\alpha\colon F \to G$.

> **Question 68.7.2.** When are two objects in the functor category isomorphic?

With this, I can make good on the last example I mentioned at the beginning:

Exercise 68.7.3. Construct the following functors:

- $\mathcal{A} \to [\mathcal{A}^{\text{op}}, \mathsf{Set}]$ by $A \mapsto H_A$, which we call $H_\bullet$.
- $\mathcal{A}^{\text{op}} \to [\mathcal{A}, \mathsf{Set}]$ by $A \mapsto H^A$, which we call $H^\bullet$.

Notice that we have opposite categories either way; even if you like H^A because it is covariant, the map $H^\bullet$ is contravariant. So for what follows, we'll prefer to use $H_\bullet$.

The main observation now is that given a category $\mathcal{A}$, $H_\bullet$ provides some *special* functors $\mathcal{A}^{\text{op}} \to \mathsf{Set}$ which are already "built" in to the category A. In light of this, we define:

Definition 68.7.4. A **presheaf** X is just a contravariant functor $\mathcal{A}^{\text{op}} \to \mathsf{Set}$. It is called **representable** if $X \cong H_A$ for some A.

In other words, when we think about representable, the question we're asking is:

> *What kind of presheaves are already "built in" to the category $\mathcal{A}$?*

One way to get at this question is: given a presheaf X and a particular H_A, we can look at the *set* of natural transformations $\alpha \colon X \implies H_A$, and see if we can learn anything about it. In fact, this set can be written explicitly:

Theorem 68.7.5 (Yoneda lemma)

Let $\mathcal{A}$ be a category, pick $A \in \mathcal{A}$, and let H_A be the contravariant Yoneda functor. Let $X \colon \mathcal{A}^{\text{op}} \to \mathsf{Set}$ be a contravariant functor. Then the map

$$\left\{ \text{Natural transformations} \quad \mathcal{A}^{\text{op}} \underset{X}{\overset{H_A}{\Downarrow \alpha}} \mathsf{Set} \right\} \to X(A)$$

defined by $\alpha \mapsto \alpha_A(\text{id}_A) \in X(A)$ is an isomorphism of Set (i.e. a bijection). Moreover, if we view both sides of the equality as functors

$$\mathcal{A}^{\text{op}} \times [\mathcal{A}^{\text{op}}, \mathsf{Set}] \to \mathsf{Set}$$

then this isomorphism is natural.

This might be startling at first sight. Here's an unsatisfying explanation why this might not be too crazy: in category theory, a rule of thumb is that "two objects of the same type that are built naturally are probably the same". You can see this theme when we defined functors and natural transformations, and even just compositions. Now to look at the set of natural transformations, we took a pair of elements $A \in \mathcal{A}$ and $X \in [\mathcal{A}^{\text{op}}, \mathsf{Set}]$ and constructed a *set* of natural transformations. Is there another way we can get a set from these two pieces of information? Yes: just look at $X(A)$. The Yoneda lemma is telling us that our heuristic still holds true here.

Some consequences of the Yoneda lemma are recorded in [**Le14**]. Since this chapter is already a bit too long, I'll just write down the statements, and refer you to [**Le14**] for the proofs.

1. As we mentioned before, $H^\bullet$ provides a functor

$$\mathcal{A} \to [\mathcal{A}^{\text{op}}, \mathsf{Set}].$$

It turns out this functor is in fact *fully faithful*; it quite literally embeds the category $\mathcal{A}$ into the functor category on the right (much like Cayley's theorem embeds every group into a permutation group).

2. If $X, Y \in \mathcal{A}$ then

$$H_X \cong H_Y \iff X \cong Y \iff H^X \cong H^Y.$$

To see why this is expected, consider $\mathcal{A} = \mathsf{Grp}$ for concreteness. Suppose A, X, Y are groups such that $H_X(A) \cong H_Y(A)$ for all A. For example,

- If $A = \mathbb{Z}$, then $|X| = |Y|$.
- If $A = \mathbb{Z}/2\mathbb{Z}$, then X and Y have the same number of elements of order 2.
- ...

Each A gives us some information on how X and Y are similar, but the whole natural isomorphism is strong enough to imply $X \cong Y$.

3. Consider the covariant forgetful functor $U : \mathsf{Grp} \to \mathsf{Set}$.[3] It can be represented by $H^{\mathbb{Z}}$, in the sense that

$$\operatorname{Hom}_{\mathsf{Grp}}(\mathbb{Z}, G) \cong U(G) \qquad \text{by} \qquad \phi \mapsto \phi(1).$$

That is, elements of G are in bijection with maps $\mathbb{Z} \to G$, determined by the image of $+1$ (or -1 if you prefer). So a representation of U was determined by looking at $\mathbb{Z}$ and picking $+1 \in U(\mathbb{Z})$.

The generalization of this is a follows: let $\mathcal{A}$ be a category and $X : \mathcal{A} \to \mathsf{Set}$ a covariant functor. Then a representation $H^A \cong X$ consists of an object $A \in \mathcal{A}$ and an element $u \in X(A)$ satisfying a certain condition. You can read this off the condition[4] if you know what the inverse map is in Theorem 68.7.5. In the above situation, $X = U$, $A = \mathbb{Z}$ and $u = \pm 1$.

§68.8 A few harder problems to think about

Problem 68A. Show that the two definitions of natural transformation (one in terms of $\mathcal{A} \times \mathbf{2} \to \mathcal{B}$ and one in terms of arrows $F(A) \xrightarrow{\alpha_A} G(A)$) are equivalent.

Problem 68B. Let $\mathcal{A}$ be the category of finite sets whose arrows are bijections between sets. For $A \in \mathcal{A}$, let $F(A)$ be the set of *permutations* of A and let $G(A)$ be the set of *orderings* on A.[5]

(a) Extend F and G to functors $\mathcal{A} \to \mathsf{Set}$.

(b) Show that $F(A) \cong G(A)$ for every A, but this isomorphism is *not* natural.

Problem 68C (Proving the Yoneda lemma)**.** In the context of Theorem 68.7.5:

(a) Prove that the map described is in fact a bijection. (To do this, you will probably have to explicitly write down the inverse map.)

(b) Prove that the bijection is indeed natural. (This is long-winded, but not difficult; from start to finish, there is only one thing you can possibly do.)

[3] Actually, you need to apply a dual version. Theorem 68.7.5 uses contravariant functor.

[4] Just for completeness, the condition is: For all $A' \in \mathcal{A}$ and $x \in X(A')$, there's a unique $f : A \to A'$ with $(Xf)(u) = x$.

[5] A permutation is a bijection $A \to A$, and an ordering is a bijection $\{1, \dots, n\} \to A$, where n is the size of A.

69 Limits in categories (TO DO)

We saw near the start of our category theory chapter the nice construction of products by drawing a bunch of arrows. It turns out that this concept can be generalized immensely, and I want to give a you taste of that here.

write introduction

To run this chapter, we follow the approach of [Le14].

§69.1 Equalizers

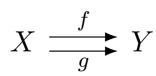
Prototypical example for this section: The equalizer of $f, g \colon X \to Y$ is the set of points with $f(x) = g(x)$.

Given two sets X and Y, and maps $X \xrightarrow{f,g} Y$, we define their **equalizer** to be

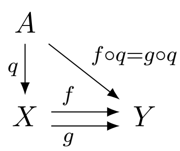
$$\{x \in X \mid f(x) = g(x)\}.$$

We would like a categorical way of defining this, too.

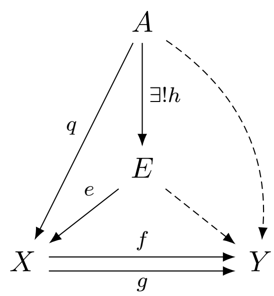
Consider two objects X and Y with two maps f and g between them. Stealing a page from [Le14], we call this a **fork**:



A cone over this fork is an object A and arrows over X and Y which make the diagram commute, like so.



Effectively, the arrow over Y is just forcing $f \circ q = g \circ q$. In any case, the **equalizer** of f and g is a "universal cone" over this fork: it is an object E and a map $E \xrightarrow{e} X$ such that for each $A \xrightarrow{q} X$ the diagram



commutes for a unique $A \xrightarrow{h} E$. In other words, any map $A \xrightarrow{q} X$ as above must factor uniquely through E. Again, the dotted arrows can be omitted, and as before equalizers may not exist. But when they do exist:

Exercise 69.1.1. If $E \xrightarrow{e} X$ and $E' \xrightarrow{e'} X$ are equalizers, show that $E \cong E'$.

Example 69.1.2 (Examples of equalizers)

(a) In Set, given $X \xrightarrow{f,g} Y$ the equalizer E can be realized as $E = \{x \mid f(x) = g(x)\}$, with the inclusion $e\colon E \hookrightarrow X$ as the morphism. As usual, by abuse we'll often just refer to E as the equalizer.

(b) Ditto in Top, Grp. One has to check that the appropriate structures are preserved (e.g. one should check that $\{\phi(g) = \psi(g) \mid g \in G\}$ is a group).

(c) In particular, given a homomorphism $\phi\colon G \to H$, the inclusion $\ker \phi \hookrightarrow G$ is an equalizer for the fork $G \to H$ by ϕ and the trivial homomorphism.

According to (c) equalizers let us get at the concept of a kernel if there is a distinguished "trivial map", like the trivial homomorphism in Grp. We'll flesh this idea out in the chapter on abelian categories.

§69.2 Pullback squares (TO DO)

write me

Great example: differentiable functions on $(-3, 1)$ and $(-1, 3)$

Example 69.2.1

§69.3 Limits

We've defined cones over discrete sets of X_i and over forks. It turns out you can also define a cone over any general **diagram** of objects and arrows; we specify a projection from A to each object and require that the projections from A commute with the arrows in the diagram. (For example, a cone over a fork is a diagram with two edges and two arrows.) If you then demand the cone be universal, you have the extremely general definition of a **limit**. As always, these are unique up to unique isomorphism. We can also define the dual notion of a **colimit** in the same way.

§69.4 A few harder problems to think about

Problem 69A* (Equalizers are monic)**.** Show that the equalizer of any fork is monic.

pushout square gives tenor product
p-adic
relative Chinese remainder theorem!!

70 Abelian categories

In this chapter I'll translate some more familiar concepts into categorical language; this will require some additional assumptions about our category, culminating in the definition of a so-called "abelian category". Once that's done, I'll be able to tell you what this "diagram chasing" thing is all about.

Throughout this chapter, "$\hookrightarrow$" will be used for monic maps and "$\twoheadrightarrow$" for epic maps.

§70.1 Zero objects, kernels, cokernels, and images

Prototypical example for this section: In Grp*, the trivial group and homomorphism are the zero objects and morphisms. If G, H are abelian then the cokernel of $\phi\colon G \to H$ is $H/\operatorname{im}\phi$.*

A **zero object** of a category is an object 0 which is both initial and terminal; of course, it's unique up to unique isomorphism. For example, in Grp the zero object is the trivial group, in Vect_k it's the zero-dimensional vector space consisting of one point, and so on.

> **Question 70.1.1.** Show that Set and Top don't have zero objects.

For the rest of this chapter, all categories will have zero objects.

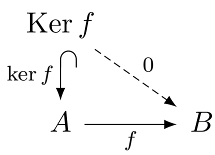
In a category $\mathcal{A}$ with zero objects, any two objects A and B thus have a distinguished morphism

$$A \to 0 \to B$$

which is called the **zero morphism** and also denoted 0. For example, in Grp this is the trivial homomorphism.

We can now define:

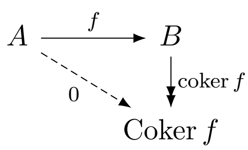
Definition 70.1.2. Consider a map $A \xrightarrow{f} B$. The **kernel** is defined as the equalizer of this map and the map $A \xrightarrow{0} B$. Thus, it's a map $\ker f\colon \operatorname{Ker} f \hookrightarrow A$ such that



commutes, and moreover any other map with the same property factors uniquely through $\operatorname{Ker} A$ (so it is universal with this property). By Problem 69A*, $\ker f$ is a monic morphism, which justifies the use of "$\hookrightarrow$".

Notice that we're using $\ker f$ to represent the map and $\operatorname{Ker} f$ to represent the object Similarly, we define the cokernel, the dual notion:

Definition 70.1.3. Consider a map $A \xrightarrow{f} B$. The **cokernel** of f is a map $\operatorname{coker} f\colon B \twoheadrightarrow \operatorname{Coker} f$ such that



commutes, and moreover any other map with the same property factors uniquely through Coker f (so it is universal with this property). Thus it is the "coequalizer" of this map and the map $A \xrightarrow{0} B$. By the dual of Problem 69A*, coker f is an epic morphism, which justifies the use of "↠".

Think of the cokernel of a map $A \xrightarrow{f} B$ as "B modulo the image of f".

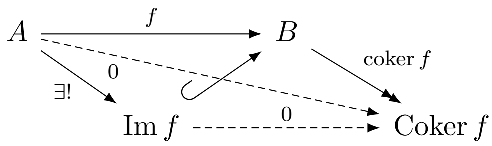
Example 70.1.4 (Cokernels)

Consider the map $\mathbb{Z}/6\mathbb{Z} \to D_{12} = \langle r, s \mid r^6 = s^2 = 1, rs = sr^{-1} \rangle$. Then the cokernel of this map in Grp is $D_{12}/\langle r \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

This doesn't always work out quite the way we want since in general the image of a homomorphism need not be normal in the codomain. Nonetheless, we can use this to define:

Definition 70.1.5. The **image** of $A \xrightarrow{f} B$ is the kernel of coker f. We denote $\operatorname{Im} f = \operatorname{Ker}(\operatorname{coker} f)$. This gives a unique map $\operatorname{im} f : A \to \operatorname{Im} f$.

When it exists, this coincides with our concrete notion of "image". Picture:



Note that by universality of $\operatorname{Im} f$, we find that there is a unique map $\operatorname{im} f : A \to \operatorname{Im} f$ that makes the entire diagram commute.

§70.2 Additive and abelian categories

Prototypical example for this section: Ab, Vect_k, *or more generally* Mod_R.

We can now define the notion of an additive and abelian category, which are the types of categories where this notion is most useful.

Definition 70.2.1. An **additive category** $\mathcal{A}$ is one such that:

- $\mathcal{A}$ has a zero object, and any two objects have a product.
- More importantly: every $\operatorname{Hom}_{\mathcal{A}}(A, B)$ forms an *abelian group* (written additively) such that composition distributes over addition:

$$(g + h) \circ f = g \circ f + h \circ f \quad \text{and} \quad f \circ (g + h) = f \circ g + f \circ h.$$

  The zero map serves as the identity element for each group.

In short:

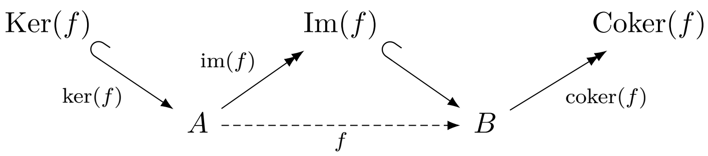
In an additive category, you can add two morphisms.

Which is the only definition that makes sense anyway, we cannot talk about elements.

Definition 70.2.2. An **abelian category** $\mathcal{A}$ is one with the additional properties that for any morphism $A \xrightarrow{f} B$,

- The kernel and cokernel exist, and
- The morphism factors through the image so that $\mathrm{im}(f)$ is epic.

So, this yields a diagram



Example 70.2.3 (Examples of abelian categories)

(a) Vect_k, Ab are abelian categories, where $f+g$ takes its usual meaning.

(b) Generalizing this, the category Mod_R of R-modules is abelian.

(c) Grp is not even additive, because there is no way to assign a commutative addition to pairs of morphisms.

From now on, you can basically forget about additive category, we will be working in abelian category.

In general, once you assume a category is abelian, all the properties you would want of these kernels, cokernels, ... that you would guess hold true. For example,

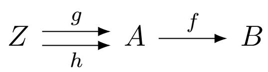
Proposition 70.2.4 (Monic $\iff$ trivial kernel)

A map $A \xrightarrow{f} B$ is monic if and only if its kernel is $0 \to A$. Dually, $A \xrightarrow{f} B$ is epic if and only if its cokernel is $B \to 0$.

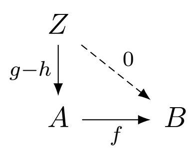
Proof. The easy direction is:

> **Exercise 70.2.5.** Show that if $A \xrightarrow{f} B$ is monic, then $0 \to A$ is a kernel. (This holds even in non-abelian categories.)

Of course, since kernels are unique up to isomorphism, monic $\implies$ 0 kernel. On the other hand, assume that $0 \to A$ is a kernel of $A \xrightarrow{f} B$. For this we can exploit the group structure of the underlying homomorphisms now. Assume the diagram



commutes. Then $(g-h)\circ f = g\circ f - h\circ f = 0$, and we've arrived at a commutative diagram.



But since $0 \to A$ is a kernel it follows that $g-h$ factors through 0, so $g-h=0 \implies g=h$, which is to say that f is monic. $\square$

Proposition 70.2.6 (Isomorphism $\iff$ monic and epic)
In an abelian category, a map is an isomorphism if and only if it is monic and epic.

Proof. Omitted. (The Mitchell embedding theorem presented later implies this anyways for most situations we care about, by looking at a small sub-category.) □

§70.3 Exact sequences

Prototypical example for this section: $0 \to G \to G \times H \to H \to 0$ *is exact.*

Exact sequences will seem exceedingly unmotivated until you learn about homology groups, which is one of the most natural places that exact sequences appear. In light of this, it might be worth trying to read the chapter on homology groups simultaneously with this one.

First, let me state the definition for groups, to motivate the general categorical definition. A sequence of groups

$$G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2 \xrightarrow{f_3} \dots \xrightarrow{f_n} G_n$$

is *exact* at G_k if the image of f_k is the kernel of f_{k+1}. We say the entire sequence is exact if it's exact at $k = 1, \dots, n-1$.

Example 70.3.1 (Exact sequences)

(a) The sequence

$$0 \to \mathbb{Z}/3\mathbb{Z} \overset{\times 5}{\hookrightarrow} \mathbb{Z}/15\mathbb{Z} \twoheadrightarrow \mathbb{Z}/5\mathbb{Z} \to 0$$

is exact. Actually, $0 \to G \hookrightarrow G \times H \twoheadrightarrow H \to 0$ is exact in general. (Here 0 denotes the trivial group.)

(b) For groups, the map $0 \to A \to B$ is exact if and only if $A \to B$ is injective.

(c) For groups, the map $A \to B \to 0$ is exact if and only if $A \to B$ is surjective.

If you look at the prototypical example, actually, a **short exact sequence** (an exact sequence of the form $0 \to A \to B \to C \to 0$) is the most natural things ever:

It's basically just an equation $C = B/A$.

Whenever you see "there is a short exact sequence $0 \to A \to B \to C \to 0$", you can mentally translate it to "$C \cong B/A$"; but there's a slight difference: A group has more structures than a number, so the sequence also contains the information of the maps — the map that identifies A with a subgroup of B, and the map that identifies C with the quotient group B/A.

Example 70.3.2 (More exact sequences)

(a) The sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \to 0$$

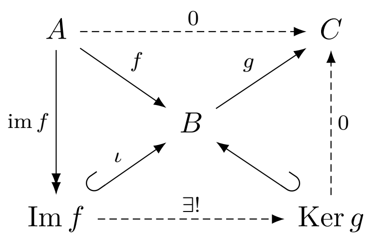
is short exact.

(b) So is

$$0 \to \mathbb{Z} \xrightarrow{\times 5} \mathbb{Z} \to \mathbb{Z}/5\mathbb{Z} \to 0.$$

As you can see, the written equation "$C \cong B/A$" is not completely accurate, the map $A \to B$ also matters in determining what C is. This also explains the common notation: the image of the map $\mathbb{Z} \xrightarrow{\times 3} \mathbb{Z}$ is usually written $3\mathbb{Z}$, thus $\mathbb{Z}/3\mathbb{Z} = \frac{\mathbb{Z}}{3\mathbb{Z}}$.

Now, we want to mimic this definition in a general *abelian* category $\mathcal{A}$. So, let's write down a criterion for when $A \xrightarrow{f} B \xrightarrow{g} C$ is exact. First, we had better have that $g \circ f = 0$, which encodes the fact that $\operatorname{im}(f) \subseteq \ker(g)$. Adding in all the relevant objects, we get the commutative diagram below.



Here the map $A \twoheadrightarrow \operatorname{Im} f$ is epic since we are assuming $\mathcal{A}$ is an abelian category. So, we have that

$$0 = (g \circ \iota) \circ \operatorname{im} f = g \circ (\iota \circ \operatorname{im} f) = g \circ f = 0$$

but since $\operatorname{im} f$ is epic, this means that $g \circ \iota = 0$. So there is a *unique* map $\operatorname{Im} f \to \operatorname{Ker} g$, and we require that this diagram commutes. In short,

Definition 70.3.3. Let $\mathcal{A}$ be an abelian category. The sequence

$$\dots \to A_{n-1} \xrightarrow{f_n} A_n \xrightarrow{f_{n+1}} A_{n+1} \to \dots$$

is **exact** at A_n if $f_n \circ f_{n+1} = 0$ and the canonical map $\operatorname{Im} f_n \to \operatorname{Ker} f_{n+1}$ is an isomorphism. The entire sequence is exact if it is exact at each A_i. (For finite sequences we don't impose condition on the very first and very last object.)

Exercise 70.3.4. Show that, as before, $0 \to A \to B$ is exact $\iff$ $A \to B$ is monic.

§70.4 The Freyd-Mitchell embedding theorem

We now introduce the Freyd-Mitchell embedding theorem, which essentially says that any abelian category can be realized as a concrete one.

Definition 70.4.1. A category is **small** if $\operatorname{obj}(\mathcal{A})$ is a set (as opposed to a class), i.e. there is a "set of all objects in $\mathcal{A}$". For example, Set is not small because there is no set of all sets.

Theorem 70.4.2 (Freyd-Mitchell embedding theorem)

Let $\mathcal{A}$ be a small abelian category. Then there exists a ring R (with 1 but possibly non-commutative) and a full, faithful, exact functor onto the category of left R-modules.

Here a functor is **exact** if it preserves exact sequences. This theorem is good because it means

You can basically forget about all the weird definitions that work in any abelian category.

Any time you're faced with a statement about an abelian category, it suffices to just prove it for a "concrete" category where injective/surjective/kernel/image/exact/etc. agree with your previous notions. A proof by this means is sometimes called *diagram chasing*.

> **Remark 70.4.3** — The "small" condition is a technical obstruction that requires the objects $\mathcal{A}$ to actually form a set. I'll ignore this distinction, because one can almost always work around it by doing enough set-theoretic technicalities.

For example, let's prove:

> **Lemma 70.4.4** (Short five lemma)
> In an abelian category, consider the commutative diagram
>
> $$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \overset{p}{\hookrightarrow} & B & \overset{q}{\twoheadrightarrow} & C & \longrightarrow & 0 \\ & & {\scriptstyle\cong}\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & {\scriptstyle\cong}\downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \underset{p'}{\hookrightarrow} & B' & \underset{q'}{\twoheadrightarrow} & C' & \longrightarrow & 0 \end{array}$$
>
> and assume the top and bottom rows are exact. If α and γ are isomorphisms, then so is β.

Proof. We prove that β is epic (with a similar proof to get monic). By the embedding theorem we can treat the category as R-modules over some R. This lets us do a so-called "diagram chase" where we move elements around the picture, using the concrete interpretation of our category as R-modules.

Let b' be an element of B'. Then $q'(b') \in C'$, and since γ is surjective, we have a c such that $\gamma(c) = b'$, and finally a $b \in B$ such that $q(b) = c$. Picture:

$$\begin{array}{ccc} b \in B & \overset{q}{\longmapsto} & c \in C \\ {\scriptstyle\beta}\,\vdots & & {\scriptstyle\cong}\downarrow{\scriptstyle\gamma} \\ b' \in B' & \overset{q'}{\longmapsto} & c' \in C' \end{array}$$

Now, it is not necessarily the case that $\beta(b) = b'$. However, since the diagram commutes we at least have that

$$q'(b') = q'(\beta(b))$$

so $b' - \beta(b) \in \operatorname{Ker} q' = \operatorname{Im} p'$, and there is an $a' \in A'$ such that $p'(a') = b' - \beta(b)$; use α now to lift it to $a \in A$. Picture:

$$\begin{array}{ccccc} a \in A & & b \in B & & \\ \downarrow & & & & \\ a' \in A' & \longmapsto & b' - \beta(b) \in B' & \longmapsto & 0 \in C \end{array}$$

Then, we have

$$\beta(b + q(a)) = \beta b + \beta p a = \beta b + p' \alpha a = \beta b + (b' - \beta b) = b'$$

so $b' \in \operatorname{Im} \beta$ which completes the proof that β' is surjective. $\square$

§70.5 Breaking long exact sequences

Prototypical example for this section: First isomorphism theorem.

In fact, it turns out that any exact sequence breaks into short exact sequences. This relies on:

Proposition 70.5.1 ("First isomorphism theorem" in abelian categories)

Let $A \xrightarrow{f} B$ be an arrow of an abelian category. Then there is an exact sequence

$$0 \to \operatorname{Ker} f \xrightarrow{\ker f} A \xrightarrow{\operatorname{im} f} \operatorname{Im} f \to 0.$$

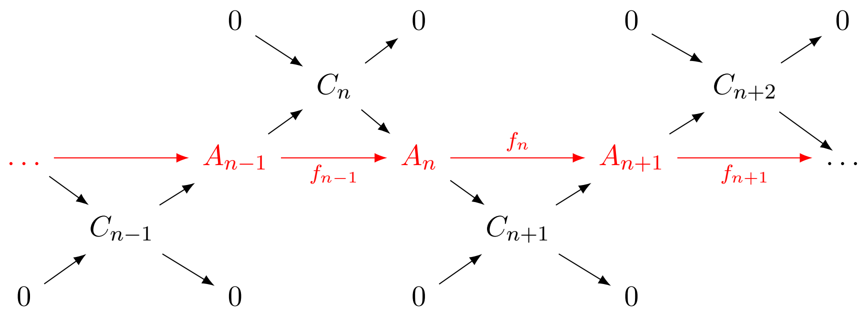
Example 70.5.2

Let's analyze this theorem in our two examples of abelian categories:

(a) In the category of abelian groups, this is basically the first isomorphism theorem.

(b) In the category Vect_k, this amounts to the rank-nullity theorem, Theorem 9.7.7.

Thus, any exact sequence can be broken into short exact sequences, as



where $C_k = \operatorname{im} f_{k-1} = \ker f_k$ for every k.

§70.6 A few harder problems to think about

Problem 70A (Four lemma)**.** In an abelian category, consider the commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{p} & B & \xrightarrow{q} & C & \xrightarrow{r} & D \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & {\scriptstyle\delta}\downarrow \\ A' & \xrightarrow[p']{} & B' & \xrightarrow[q']{} & C' & \xrightarrow[r']{} & D' \end{array}$$

where the first and second rows are exact. Prove that if α is epic, and β and δ are monic, then γ is monic.

Problem 70B (Five lemma)**.** In an abelian category, consider the commutative diagram

$$\begin{array}{ccccccccc} A & \xrightarrow{p} & B & \xrightarrow{q} & C & \xrightarrow{r} & D & \xrightarrow{s} & E \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow{\scriptstyle\cong} & & {\scriptstyle\gamma}\downarrow & & {\scriptstyle\delta}\downarrow{\scriptstyle\cong} & & {\scriptstyle\varepsilon}\downarrow \\ A' & \xrightarrow[p']{} & B' & \xrightarrow[q']{} & C' & \xrightarrow[r']{} & D' & \xrightarrow[s']{} & E' \end{array}$$

where the two rows are exact, β and δ are isomorphisms, α is epic, and ε is monic. Prove that γ is an isomorphism.

Problem 70C* (Snake lemma)**.** In an abelian category, consider the diagram

$$\begin{array}{ccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c & & \\ 0 & \longrightarrow & A' & \xhookrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & & \end{array}$$

where the first and second rows are exact sequences. Prove that there is an exact sequence

$$\operatorname{Ker} a \to \operatorname{Ker} b \to \operatorname{Ker} c \to \operatorname{Coker} a \to \operatorname{Coker} b \to \operatorname{Coker} c.$$

Problem 70D (An additive category that is not abelian)**.** Consider a category, where:

- the objects are pairs of abelian groups (B, A) where A is a subgroup of B.
- the morphisms $(B, A) \to (B', A')$ are maps $f : B \to B'$ where $f^{\text{img}}(A) \subseteq A'$.

(You can think of this similar to the PairTop category, seen in Chapter 73. We use abelian groups here to make the category additive.)

This category can be equivalently viewed as the category of short exact sequences $0 \to A \to B \to B/A \to 0$ of abelian groups.

Show that the arrow $(X, 0) \to (X, X)$ is monic and epic, but not an isomorphism. Conclude that the category is not abelian.