XVI

Algebraic Topology I: Homotopy

Part XVI: Contents

64 Some topological constructions

In this short chapter we briefly describe some common spaces and constructions in topology that we haven't yet discussed.

§64.1 Spheres

Recall that

$$S^n = \left\{(x_0, \dots, x_n) \mid x_0^2 + \cdots + x_n^2 = 1\right\} \subset \mathbb{R}^{n+1}$$

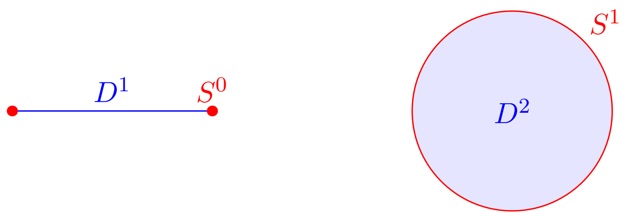
is the surface of an n-sphere while

$$D^{n+1} = \left\{(x_0, \dots, x_n) \mid x_0^2 + \cdots + x_n^2 \le 1\right\} \subset \mathbb{R}^{n+1}$$

is the corresponding *closed ball* (So for example, D^2 is a disk in a plane while S^1 is the unit circle.)

> **Exercise 64.1.1.** Show that the open ball $D^n \setminus S^{n-1}$ is homeomorphic to $\mathbb{R}^n$.

In particular, S^0 consists of two points, while D^1 can be thought of as the interval $[-1, 1]$.



§64.2 Quotient topology

Prototypical example for this section: $D^n/S^{n-1} = S^n$, *or the torus.*

Given a space X, we can *identify* some of the points together by any equivalence relation $\sim$; for an $x \in X$ we denote its equivalence class by $[x]$. Geometrically, this is the space achieved by welding together points equivalent under $\sim$.

Formally,

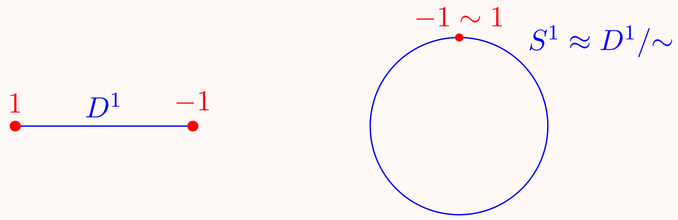
Definition 64.2.1. Let X be a topological space, and $\sim$ an equivalence relation on the points of X. Then $X/\sim$ is the space whose

- Points are equivalence classes of X, and
- $U \subseteq X/\sim$ is open if and only if $\{x \in X \text{ such that } [x] \in U\}$ is open in X.

As far as I can tell, this definition is mostly useless for intuition, so here are some examples.

Example 64.2.2 (Interval modulo endpoints)

Suppose we take $D^1 = [-1, 1]$ and quotient by the equivalence relation which identifies the endpoints -1 and 1. (Formally, $x \sim y \iff (x = y)$ or $\{x, y\} = \{-1, 1\}$.) In that case, we simply recover S^1:



Observe that a small open neighborhood around $-1 \sim 1$ in the quotient space corresponds to two half-intervals at -1 and 1 in the original space D^1. This should convince you the definition we gave is the right one.

Example 64.2.3 (More quotient spaces)

Convince yourself that:

- Generalizing the previous example, D^n modulo its boundary S^{n-1} is S^n.
- Given a square $ABCD$, suppose we identify segments AB and DC together. Then we get a cylinder. (Think elementary school, when you would tape up pieces of paper together to get cylinders.)
- In the previous example, if we also identify BC and DA together, then we get a torus. (Imagine taking our cylinder and putting the two circles at the end together.)
- Let $X = \mathbb{R}$, and let $x \sim y$ if $y - x \in \mathbb{Z}$. Then $X/\sim$ is S^1 as well.

One special case that we did above:

Definition 64.2.4. Let $A \subseteq X$. Consider the equivalence relation which identifies all the points of A with each other while leaving all remaining points inequivalent. (In other words, $x \sim y$ if $x = y$ or $x, y \in A$.) Then the resulting quotient space is denoted X/A.

So in this notation,

$$D^n/S^{n-1} = S^n.$$

Abuse of Notation 64.2.5. Note that I'm deliberately being sloppy, and saying "$D^n/S^{n-1} = S^n$" or "D^n/S^{n-1} *is* S^n", when I really ought to say "D^n/S^{n-1} is homeomorphic to S^n". This is a general theme in mathematics: objects which are homoeomorphic/isomorphic/etc. are generally not carefully distinguished from each other.

Example 64.2.6 (Weirder quotient spaces)

If the subset A is not closed in X, X/A would be quite weird.
For instance, let $X = \mathbb{R}$ and $A = (0, 1)$. Then the space X/A consists of the points $(-\infty, 0] \cup \{A/A\} \cup [1, \infty)$. Here, the points 0 and A/A are different; yet every open

set that contains 0, also contains A/A.
We say this space X/A is not Hausdorff.

§64.3 Product topology

Prototypical example for this section: $\mathbb{R} \times \mathbb{R}$ *is* $\mathbb{R}^2$, $S^1 \times S^1$ *is the torus.*

Definition 64.3.1. Given topological spaces X and Y, the **product topology** on $X \times Y$ is the space whose

- Points are pairs (x, y) with $x \in X$, $y \in Y$, and
- Topology is given as follows: the *basis* of the topology for $X \times Y$ is $U \times V$, for $U \subseteq X$ open and $V \subseteq Y$ open.

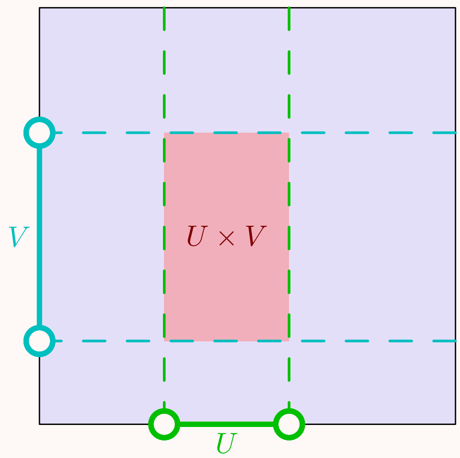
Remark 64.3.2 — It is not hard to show that, in fact, one need only consider basis elements for U and V. That is to say,

$$\{U \times V \mid U, V \text{ basis elements for } X, Y\}$$

is also a basis for $X \times Y$.
We really do need to fiddle with the basis: in $\mathbb{R} \times \mathbb{R}$, an open unit disk better be open, despite not being of the form $U \times V$.

This does exactly what you think it would.

Example 64.3.3 (The unit square)
Let $X = [0, 1]$ and consider $X \times X$. We of course expect this to be the unit square. Pictured below is an open set of $X \times X$ in the basis.



Exercise 64.3.4. Convince yourself this basis gives the same topology as the product metric on $X \times X$. So this is the "right" definition.

Example 64.3.5 (More product spaces)

(a) $\mathbb{R} \times \mathbb{R}$ is the Euclidean plane.

(b) $S^1 \times [0,1]$ is a cylinder.

(c) $S^1 \times S^1$ is a torus! (Why?)

§64.4 Disjoint union and wedge sum

Prototypical example for this section: $S^1 \vee S^1$ is the figure eight.

The disjoint union of two spaces is geometrically exactly what it sounds like: you just imagine the two spaces side by side. For completeness, here is the formal definition.

Definition 64.4.1. Let X and Y be two topological spaces. The **disjoint union**, denoted $X \amalg Y$, is defined by

- The points are the disjoint union $X \amalg Y$, and
- A subset $U \subseteq X \amalg Y$ is open if and only if $U \cap X$ and $U \cap Y$ are open.

Exercise 64.4.2. Show that the disjoint union of two nonempty spaces is disconnected.

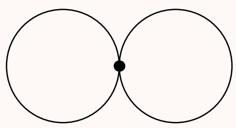
More interesting is the wedge sum, where two topological spaces X and Y are fused together only at a single base point.

Definition 64.4.3. Let X and Y be topological spaces, and $x_0 \in X$ and $y_0 \in Y$ be points. We define the equivalence relation $\sim$ by declaring $x_0 \sim y_0$ only. Then the **wedge sum** of two spaces is defined as

$$X \vee Y = (X \amalg Y)/\sim.$$

Example 64.4.4 ($S^1 \vee S^1$ is a figure eight)

Let $X = S^1$ and $Y = S^1$, and let $x_0 \in X$ and $y_0 \in Y$ be any points. Then $X \vee Y$ is a "figure eight": it is two circles fused together at one point.



Abuse of Notation 64.4.5. We often don't mention x_0 and y_0 when they are understood (or irrelevant). For example, from now on we will just write $S^1 \vee S^1$ for a figure eight.

Remark 64.4.6 — Annoyingly, in LaTeX `\wedge` gives $\wedge$ instead of $\vee$ (which is `\vee`). So this really should be called the "vee product", but too late.

§64.5 CW complexes

Using this construction, we can start building some spaces. One common way to do so is using a so-called **CW complex**. Intuitively, a CW complex is built as follows:

- Start with a set of points X^0.
- Define X^1 by taking some line segments (copies of D^1) and fusing the endpoints (copies of S^0) onto X^0.
- Define X^2 by taking copies of D^2 (a disk) and welding its boundary (a copy of S^1) onto X^1.
- Repeat inductively up until a finite stage n; we say X is **n-dimensional**.

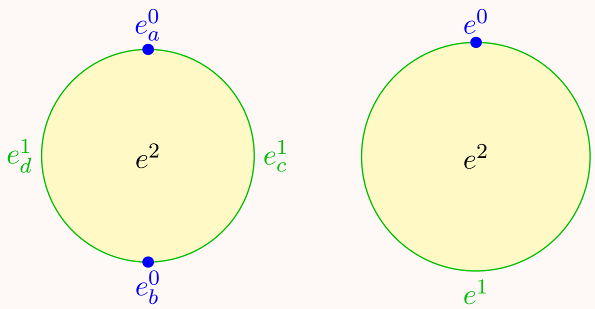
The resulting space X is the CW-complex. The set X^k is called the **k-skeleton** of X. Each D^k is called a **k-cell**; it is customary to denote it by e^k_α where α is some index. We say that X is **finite** if only finitely many cells were used.

Abuse of Notation 64.5.1. Technically, most sources (like [**Ha02**]) allow one to construct infinite-dimensional CW complexes. We will not encounter any such spaces in the Napkin.

Example 64.5.2 (D^2 with $2+2+1$ and $1+1+1$ cells)

(a) First, we start with X^0 having two points e^0_a and e^0_b. Then, we join them with two 1-cells D^1 (green), call them e^1_c and e^1_d. The endpoints of each 1-cell (the copy of S^0) get identified with distinct points of X^0; hence $X^1 \cong S^1$. Finally, we take a single 2-cell e^2 (yellow) and weld it in, with its boundary fitting into the copy of S^1 that we just drew. This gives the figure on the left.

(b) In fact, one can do this using just $1+1+1=3$ cells. Start with X^0 having a single point e^0. Then, use a single 1-cell e^1, fusing its two endpoints into the single point of X^0. Then, one can fit in a copy of S^1 as before, giving D^2 as on the right.



Example 64.5.3 (S^n as a CW complex)

(a) One can obtain S^n (for $n \ge 1$) with just two cells. Namely, take a single point e^0 for X^0, and to obtain S^n take D^n and weld its entire boundary into e^0.

We already saw this example in the beginning with $n=2$, when we saw that the sphere S^2 was the result when we fuse the boundary of a disk D^2 together.

(b) Alternatively, one can do a "hemisphere" construction, by constructing S^n inductively using two cells in each dimension. So S^0 consists of two points, then S^1 is obtained by joining these two points by two segments (1-cells), and S^2 is obtained by gluing two hemispheres (each a 2-cell) with S^1 as its equator.

Definition 64.5.4. Formally, for each k-cell e^k_α we want to add to X^k, we take its boundary S^{k-1}_α and weld it onto X^{k-1} via an **attaching map** $S^{k-1}_\alpha \to X^{k-1}$. Then

$$X^k = \left(X^{k-1} \amalg \left(\coprod_\alpha e^k_\alpha \right) \right) /\sim$$

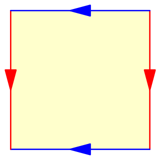
where $\sim$ identifies each boundary point of e^k_α with its image in X^{k-1}.

§64.6 The torus, Klein bottle, $\mathbb{RP}^n$, $\mathbb{CP}^n$

We now present four of the most important examples of CW complexes.

§64.6.i The torus

The **torus** can be formed by taking a square and identifying the opposite edges in the same direction: if you walk off the right edge, you re-appear at the corresponding point in on the left edge. (Think *Asteroids* from Atari!)



Thus the torus is $(\mathbb{R}/\mathbb{Z})^2 \cong S^1 \times S^1$.

Note that all four corners get identified together to a single point. One can realize the torus in 3-space by treating the square as a sheet of paper, taping together the left and right (red) edges to form a cylinder, then bending the cylinder and fusing the top and bottom (blue) edges to form the torus.

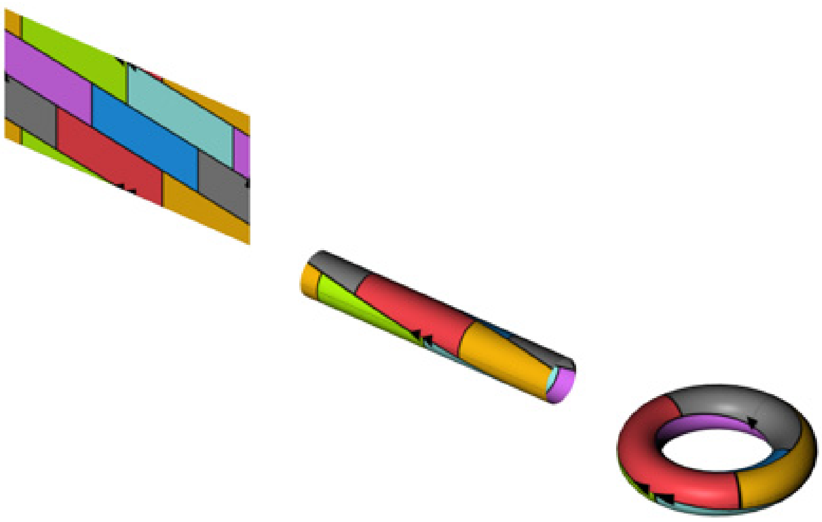
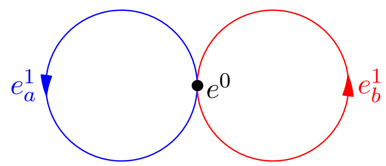


Image from [To]

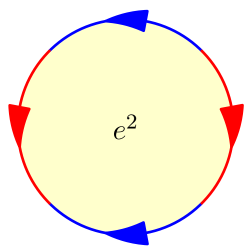
The torus can be realized as a CW complex with

- A 0-skeleton consisting of a single point,

- A 1-skeleton consisting of two 1-cells e_a^1, e_b^1, and



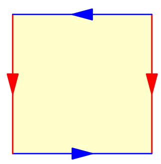
- A 2-skeleton with a single 2-cell e^2, whose circumference is divided into four parts, and welded onto the 1-skeleton "via $aba^{-1}b^{-1}$". This means: wrap a quarter of the circumference around e_a^1, then another quarter around e_b^1, then the third quarter around e_a^1 but in the opposite direction, and the fourth quarter around e_b^1 again in the opposite direction as before.



We say that $aba^{-1}b^{-1}$ is the **attaching word**; this shorthand will be convenient later on.

§64.6.ii The Klein bottle

The **Klein bottle** is defined similarly to the torus, except one pair of edges is identified in the opposite manner, as shown.



Unlike the torus one cannot realize this in 3-space without self-intersecting. One can tape together the red edges as before to get a cylinder, but to then fuse the resulting blue circles in opposite directions is not possible in 3D. Nevertheless, we often draw a picture in 3-dimensional space in which we tacitly allow the cylinder to intersect itself.

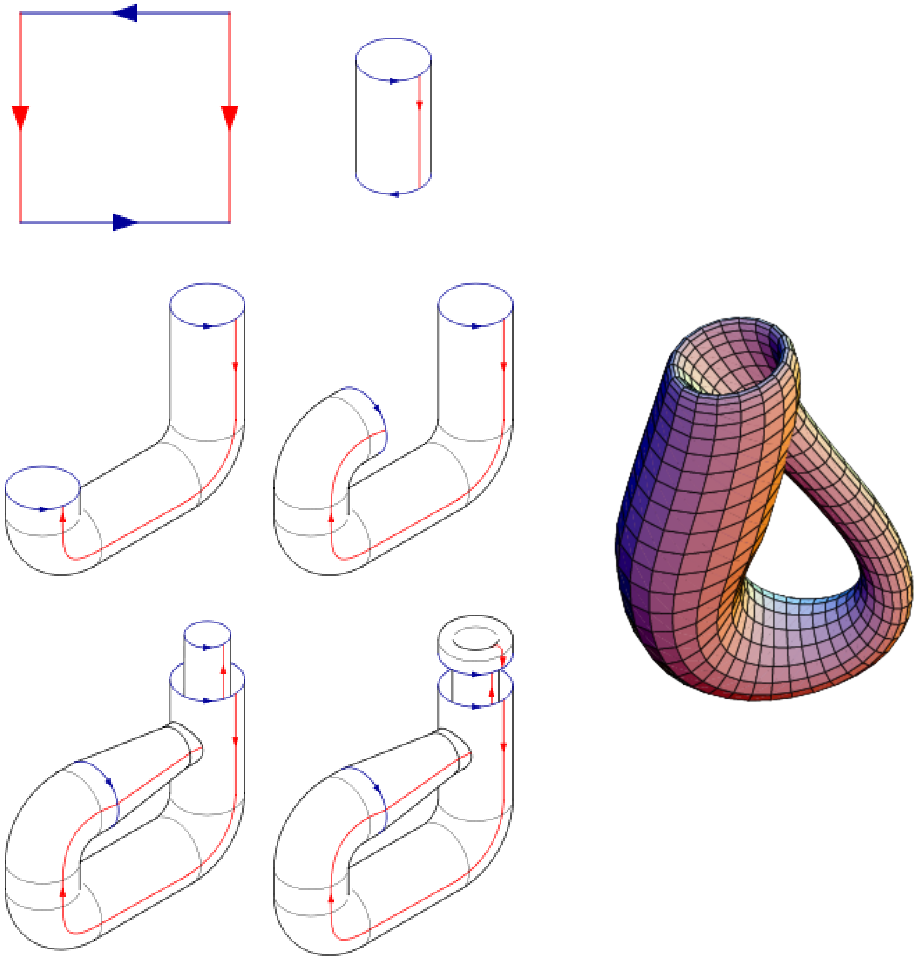


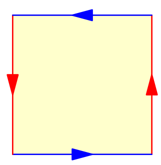
Image from [**In**; **Fr**]

Like the torus, the Klein bottle is realized as a CW complex with

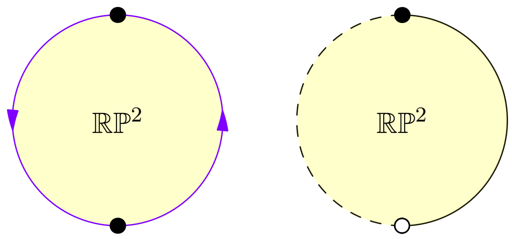
- One 0-cell,
- Two 1-cells e^1_a and e^1_b, and
- A single 2-cell attached this time via the word $abab^{-1}$.

§64.6.iii Real projective space

Let's start with $n = 2$. The space $\mathbb{RP}^2$ is obtained if we reverse both directions of the square from before, as shown.



However, once we do this the fact that the original polygon is a square is kind of irrelevant; we can combine a red and blue edge to get the single purple edge. Equivalently, one can think of this as a circle with half its circumference identified with the other half:



The resulting space should be familiar to those of you who do projective (Euclidean) geometry. Indeed, there are several possible geometric interpretations:

- One can think of $\mathbb{RP}^2$ as the set of lines through the origin in $\mathbb{R}^3$, with each line being a point in $\mathbb{RP}^2$.

  Of course, we can identify each line with a point on the unit sphere S^2, except for the property that two antipodal points actually correspond to the same line, so that $\mathbb{RP}^2$ can be almost thought of as "half a sphere". Flattening it gives the picture above.

- Imagine $\mathbb{R}^2$, except augmented with "points at infinity". This means that we add some points "infinitely far away", one for each possible direction of a line. Thus in $\mathbb{RP}^2$, any two lines indeed intersect (at a Euclidean point if they are not parallel, and at a point at infinity if they do).

  This gives an interpretation of $\mathbb{RP}^2$, where the boundary represents the *line at infinity* through all of the points at infinity. Here we have used the fact that $\mathbb{R}^2$ and interior of D^2 are homeomorphic.

> **Exercise 64.6.1.** Observe that these formulations are equivalent by considering the plane $z = 1$ in $\mathbb{R}^3$, and intersecting each line in the first formulation with this plane.

We can also express $\mathbb{RP}^2$ using coordinates: it is the set of triples $(x : y : z)$ of real numbers not all zero up to scaling, meaning that

$$(x : y : z) = (\lambda x : \lambda y : \lambda z)$$

for any $\lambda \neq 0$. Using the "lines through the origin in $\mathbb{R}^3$" interpretation makes it clear why this coordinate system gives the right space. The points at infinity are those with $z = 0$, and any point with $z \neq 0$ gives a Cartesian point since

$$(x : y : z) = \left(\frac{x}{z} : \frac{y}{z} : 1\right)$$

hence we can think of it as the Cartesian point $(\frac{x}{z}, \frac{y}{z})$.

In this way we can actually define **real-projective n-space**, $\mathbb{RP}^n$ for any n, as either

(i) The set of lines through the origin in $\mathbb{R}^{n+1}$,

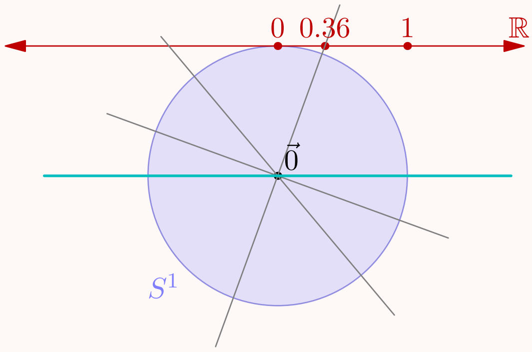
(ii) Using $n + 1$ coordinates as above, or

(iii) As $\mathbb{R}^n$ augmented with points at infinity, which themselves form a copy of $\mathbb{RP}^{n-1}$.

As a possibly helpful example, we give all three pictures of $\mathbb{RP}^1$.

Example 64.6.2 (Real projective 1-Space)

$\mathbb{RP}^1$ can be thought of as S^1 modulo the relation the antipodal points are identified. Projecting onto a tangent line, we see that we get a copy of $\mathbb{R}$ plus a single point at infinity, corresponding to the parallel line (drawn in cyan below).



Thus, the points of $\mathbb{RP}^1$ have two forms:

- $(x : 1)$, which we think of as $x \in \mathbb{R}$ (in dark red above), and
- $(1 : 0)$, which we think of as $1/0 = \infty$, corresponding to the cyan line above.

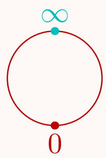
So, we can literally write

$$\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}.$$

Note that $\mathbb{RP}^1$ is also the boundary of $\mathbb{RP}^2$. In fact, note also that topologically we have

$$\mathbb{RP}^1 \cong S^1$$

since it is the "real line with endpoints fused together".



Since $\mathbb{RP}^n$ is just "$\mathbb{R}^n$ (or D^n) with $\mathbb{RP}^{n-1}$ as its boundary", we can construct $\mathbb{RP}^n$ as a CW complex inductively. Note that $\mathbb{RP}^n$ thus consists of **one cell in each dimension**.

Example 64.6.3 ($\mathbb{RP}^n$ as a cell complex)

(a) $\mathbb{RP}^0$ is a single point.

(b) $\mathbb{RP}^1 \cong S^1$ is a circle, which as a CW complex is a 0-cell plus a 1-cell.

(c) $\mathbb{RP}^2$ can be formed by taking a 2-cell and wrapping its perimeter twice around a copy of $\mathbb{RP}^1$.

§64.6.iv Complex projective space

The **complex projective space** $\mathbb{CP}^n$ is defined like $\mathbb{RP}^n$ with coordinates, i.e.

$$(z_0 : z_1 : \cdots : z_n)$$

under scaling; this time z_i are complex. As before, $\mathbb{CP}^n$ can be thought of as $\mathbb{C}^n$ augmented with some points at infinity (corresponding to $\mathbb{CP}^{n-1}$).

Example 64.6.4 (Complex projective space)

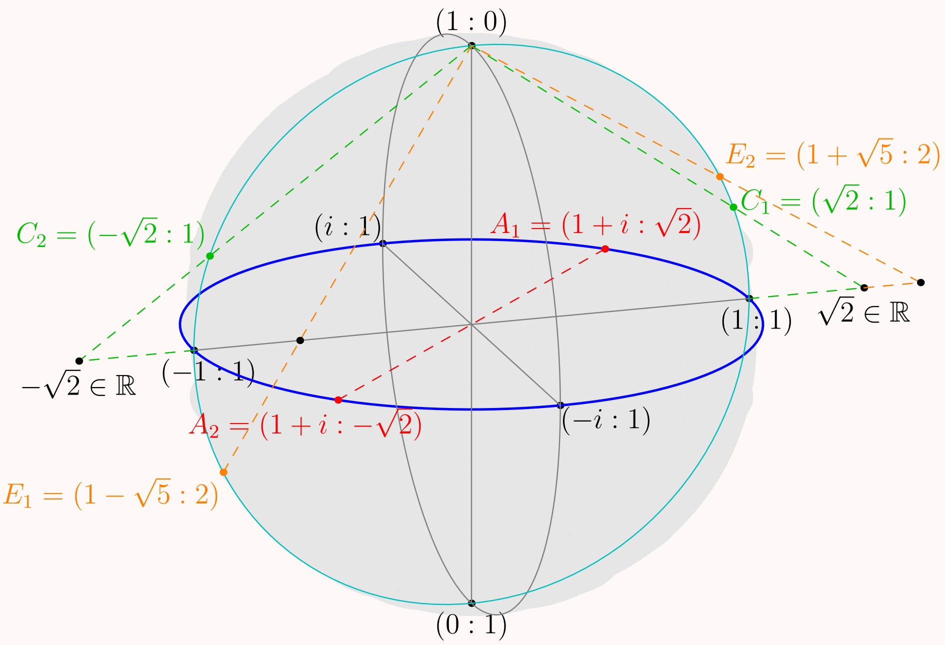
(a) $\mathbb{CP}^0$ is a single point.

(b) $\mathbb{CP}^1$ is $\mathbb{C}$ plus a single point at infinity ("complex infinity" if you will). That means as before we can think of $\mathbb{CP}^1$ as

$$\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}.$$

So, imagine taking the complex plane and then adding a single point to encompass the entire boundary. The result is just sphere S^2.

Here is a picture of $\mathbb{CP}^1$ with its coordinate system, the **Riemann sphere**.



Remark 64.6.5 (For Euclidean geometers) — You may recognize that while $\mathbb{RP}^2$ is the setting for projective geometry, inversion about a circle is done in $\mathbb{CP}^1$ instead. When one does an inversion sending generalized circles to generalized circles, there is only one point at infinity: this is why we work in $\mathbb{CP}^n$.

Like $\mathbb{RP}^n$, $\mathbb{CP}^n$ is a CW complex, built inductively by taking $\mathbb{C}^n$ and welding its boundary onto $\mathbb{CP}^{n-1}$ The difference is that as topological spaces,

$$\mathbb{C}^n \cong \mathbb{R}^{2n} \cong D^{2n}.$$

Thus, we attach the cells D^0, D^2, D^4 and so on inductively to construct $\mathbb{CP}^n$. Thus we see that

$\mathbb{CP}^n$ consists of one cell in each *even* dimension.

§64.7 A few harder problems to think about

Problem 64A. Show that a space X is Hausdorff if and only if the diagonal $\{(x, x) \mid x \in X\}$ is closed in the product space $X \times X$.

Problem 64B. Realize the following spaces as CW complexes:

(a) Möbius strip.

(b) $\mathbb{R}$.

(c) $\mathbb{R}^n$.

Problem 64C†. Show that a finite CW complex is compact.

65 Fundamental groups

Topologists can't tell the difference between a coffee cup and a doughnut. So how do you tell *anything* apart?

This is a very hard question to answer, but one way we can try to answer it is to find some *invariants* of the space. To draw on the group analogy, two groups are clearly not isomorphic if, say, they have different orders, or if one is simple and the other isn't, etc. We'd like to find some similar properties for topological spaces so that we can actually tell them apart.

Two such invariants for a space X are

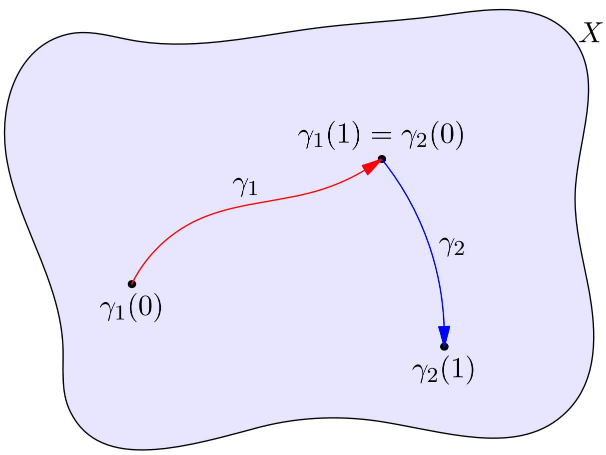
- Defining homology groups $H_1(X)$, $H_2(X)$, ...
- Defining homotopy groups $\pi_1(X)$, $\pi_2(X)$, ...

Homology groups are hard to define, but in general easier to compute. Homotopy groups are easier to define but harder to compute.

This chapter is about the fundamental group π_1.

§65.1 Fusing paths together

Recall that a *path* in a space X is a function $[0, 1] \to X$. Suppose we have paths γ_1 and γ_2 such that $\gamma_1(1) = \gamma_2(0)$. We'd like to fuse[1] them together to get a path $\gamma_1 * \gamma_2$. Easy, right?



We unfortunately do have to hack the definition a tiny bit. In an ideal world, we'd have a path $\gamma_1 \colon [0, 1] \to X$ and $\gamma_2 \colon [1, 2] \to X$ and we could just merge them together to get $\gamma_1 * \gamma_2 \colon [0, 2] \to X$. But the "2" is wrong here. The solution is that we allocate $[0, \frac{1}{2}]$ for the first path and $[\frac{1}{2}, 1]$ for the second path; we run "twice as fast".

[1] Almost everyone else in the world uses "gluing" to describe this and other types of constructs. But I was traumatized by Elmer's glue when I was in high school because I hated the stupid "make a poster" projects and hated having to use glue on them. So I refuse to talk about "gluing" paths together, referring instead to "fusing" them together, which sounds cooler anyways.

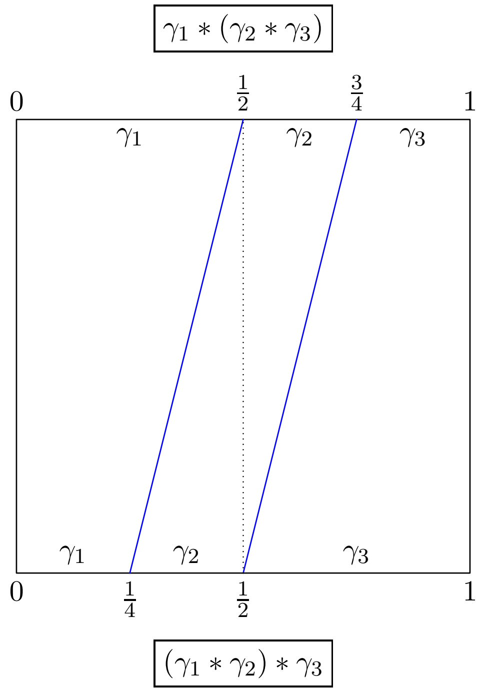
Definition 65.1.1. Given two paths $\gamma_1, \gamma_2 \colon [0,1] \to X$ such that $\gamma_1(1) = \gamma_2(0)$, we define a path $\gamma_1 * \gamma_2 \colon [0,1] \to X$ by

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(2t) & 0 \le t \le \frac{1}{2} \\ \gamma_2(2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$

This hack unfortunately reveals a second shortcoming: this "product" is not associative. If we take $(\gamma_1 * \gamma_2) * \gamma_3$ for some suitable paths, then $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ are the times allocated for γ_1, γ_2, γ_3.

Question 65.1.2. What are the times allocated for $\gamma_1 * (\gamma_2 * \gamma_3)$?

But I hope you'll agree that even though this operation isn't associative, the reason it fails to be associative is kind of stupid. It's just a matter of how fast we run in certain parts.



So as long as we're fusing paths together, we probably don't want to think of $[0,1]$ itself too seriously. And so we only consider everything up to (path) homotopy equivalence. (Recall that two paths α and β are homotopic if there's a path homotopy $F \colon [0,1]^2 \to X$ between them, which is a continuous deformation from α to β.) It is definitely true that

$$(\gamma_1 * \gamma_2) * \gamma_3 \simeq \gamma_1 * (\gamma_2 * \gamma_3)\,.$$

It is also true that if $\alpha_1 \simeq \alpha_2$ and $\beta_1 \simeq \beta_2$ then $\alpha_1 * \beta_1 \simeq \alpha_2 * \beta_2$.

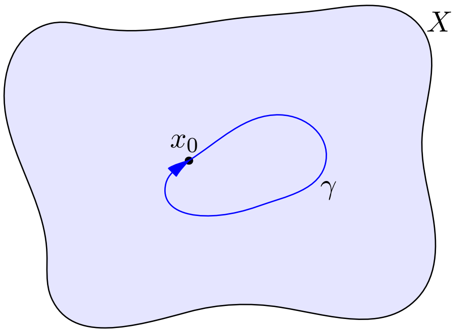
Naturally, homotopy is an equivalence relation, so paths γ lives in some "homotopy type", the equivalence classes under $\simeq$. We'll denote this $[\gamma]$. Then it makes sense to talk about $[\alpha] * [\beta]$. Thus, **we can think of $*$ as an operation on homotopy classes**.

§65.2 Fundamental groups

Prototypical example for this section: $\pi_1(\mathbb{R}^2)$ *is trivial and* $\pi_1(S^1) \cong \mathbb{Z}$.

At this point I'm a little annoyed at keeping track of endpoints, so now I'm going to specialize to a certain type of path.

Definition 65.2.1. A **loop** is a path with $\gamma(0) = \gamma(1)$.



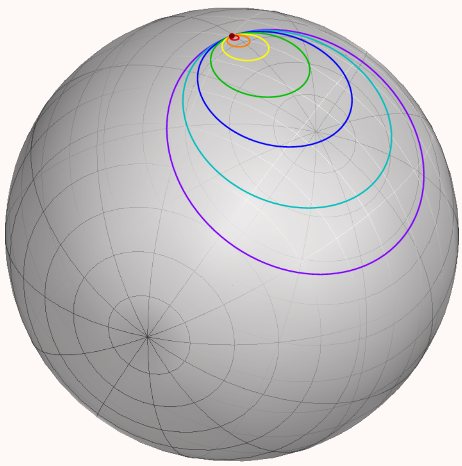
Hence if we restrict our attention to paths starting at a single point x_0, then we can stop caring about endpoints and start-points, since everything starts and stops at x_0. We even have a very canonical loop: the "do-nothing" loop[2] given by standing at x_0 the whole time.

Definition 65.2.2. Denote the trivial "do-nothing loop" by 1. A loop γ is **nulhomotopic** if it is homotopic to 1; i.e. $\gamma \simeq 1$.

For homotopy of loops, you might visualize "reeling in" the loop, contracting it to a single point.

Example 65.2.3 (Loops in S^2 are nulhomotopic)

As the following picture should convince you, every loop in the simply connected space S^2 is nulhomotopic.



(Starting with the purple loop, we contract to the red-brown point.)

Hence to show that spaces are simply connected it suffices to understand the loops of that space. We are now ready to provide:

[2]Fatty.

Definition 65.2.4. The **fundamental group** of X with basepoint x_0, denoted $\pi_1(X, x_0)$, is the set of homotopy classes

$$\{[\gamma] \mid \gamma \text{ a loop at } x_0\}$$

equipped with $*$ as a group operation.

It might come as a surprise that this has a group structure. For example, what is the inverse? Let's define it now.

Definition 65.2.5. Given a path $\alpha\colon [0,1] \to X$ we can define a path $\overline{\alpha}$

$$\overline{\alpha}(t) = \alpha(1-t).$$

In effect, this "runs α backwards". Note that $\overline{\alpha}$ starts at the endpoint of α and ends at the starting point of α.

> **Exercise 65.2.6.** Show that for any path α, $\alpha * \overline{\alpha}$ is homotopic to the "do-nothing" loop at $\alpha(0)$. (Draw a picture.)

Let's check it.

Proof that this is a group structure. Clearly $*$ takes two loops at x_0 and spits out a loop at x_0. We also already took the time to show that $*$ is associative. So we only have to check that (i) there's an identity, and (ii) there's an inverse.

- We claim that the identity is the "do-nothing" loop 1 we described above. The reader can check that for any γ,

$$\gamma \simeq \gamma * 1 \simeq 1 * \gamma.$$

- For a loop γ, recall again we define its "backwards" loop $\overline{\gamma}$ by

$$\overline{\gamma}(t) = \gamma(1-t).$$

  Then we have $\gamma * \overline{\gamma} = \overline{\gamma} * \gamma = 1$.

Hence $\pi_1(X, x_0)$ is actually a group. □

Before going any further I had better give some examples.

Example 65.2.7 (Examples of fundamental groups)

Note that proving the following results is not at all trivial. For now, just try to see intuitively why the claimed answer "should" be correct.

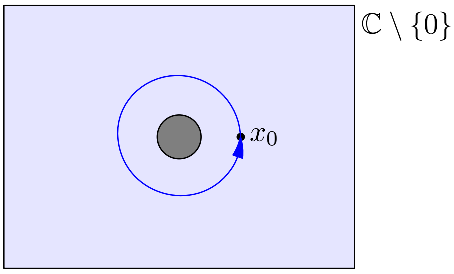
(a) The fundamental group of $\mathbb{C}$ is the trivial group: in the plane, every loop is nulhomotopic. (Proof: imagine it's a piece of rope and reel it in.)

(b) On the other hand, the fundamental group of $\mathbb{C} \setminus \{0\}$ (meteor example from earlier) with any base point is actually $\mathbb{Z}$! We won't be able to prove this for a while, but essentially a loop is determined by the number of times that it winds around the origin – these are so-called *winding numbers*. Think about it!

(c) Similarly, we will soon show that the fundamental group of S^1 (the boundary of the unit circle) is $\mathbb{Z}$.

Officially, I also have to tell you what the base point is, but by symmetry in these examples, it doesn't matter.

Here is the picture for $\mathbb{C} \setminus \{0\}$, with the hole exaggerated as the meteor from Section 7.7.



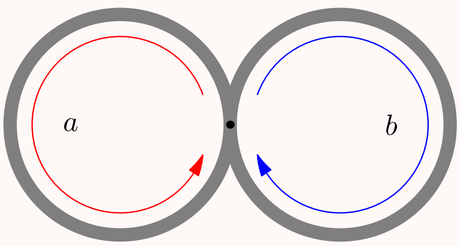
Question 65.2.8. Convince yourself that the fundamental group of S^1 is $\mathbb{Z}$, and understand why we call these "winding numbers". (This will be the most important example of a fundamental group in later chapters, so it's crucial you figure it out now.)

Example 65.2.9 (The figure eight)

Consider a figure eight $S^1 \vee S^1$, and let x_0 be the center. Then

$$\pi_1(S^1 \vee S^1, x_0) \cong \langle a, b \rangle$$

is the *free group* generated on two letters. The idea is that one loop of the eight is a, and the other loop is b, so we expect π_1 to be generated by this loop a and b (and its inverses $\overline{a}$ and $\overline{b}$). These loops don't talk to each other.

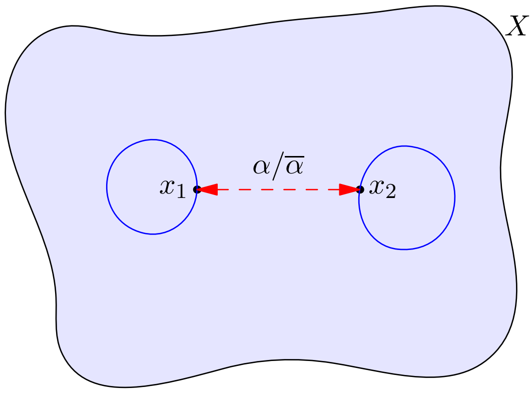


Recall that in graph theory, we usually assume our graphs are connected, since otherwise we can just consider every connected component separately. Likewise, we generally want to restrict our attention to path-connected spaces, since if a space isn't path-connected then it can be broken into a bunch of "path-connected components". (Can you guess how to define this?) Indeed, you could imagine a space X that consists of the objects on my desk (but not the desk itself): π_1 of my phone has nothing to do with π_1 of my mug. They are just totally disconnected, both figuratively and literally.

But on the other hand we claim that in a path-connected space, the groups are very related!

Theorem 65.2.10 (Fundamental groups don't depend on basepoint)
Let X be a path-connected space. Then for any $x_1 \in X$ and $x_2 \in X$, we have
$$\pi_1(X, x_1) \cong \pi_1(X, x_2).$$

Before you read the proof, see if you can guess the isomorphism based just on the picture below.



Proof. Let α be any path from x_1 to x_2 (possible by path-connectedness), and let $\overline{\alpha}$ be its reverse. Then we can construct a map
$$\pi_1(X, x_1) \to \pi_1(X, x_2) \text{ by } [\gamma] \mapsto [\overline{\alpha} * \gamma * \alpha].$$
In other words, given a loop γ at x_1, we can start at x_2, follow $\overline{\alpha}$ to x_1, run γ, then run along α home to x_2. Hence this is a map which builds a loop of $\pi_1(X, x_2)$ from every loop at $\pi_1(X, x_1)$. It is a *homomorphism* of the groups just because
$$(\overline{\alpha} * \gamma_1 * \alpha) * (\overline{\alpha} * \gamma_2 * \alpha) = \overline{\alpha} * \gamma_1 * \gamma_2 * \alpha$$
as $\alpha * \overline{\alpha}$ is nulhomotopic.

Similarly, there is a homomorphism
$$\pi_1(X, x_2) \to \pi_1(X, x_1) \text{ by } [\gamma] \mapsto [\alpha * \gamma * \overline{\alpha}].$$
As these maps are mutual inverses, it follows they must be isomorphisms. End of story. □

This is a bigger reason why we usually only care about path-connected spaces.

Abuse of Notation 65.2.11. For a path-connected space X we will often abbreviate $\pi_1(X, x_0)$ to just $\pi_1(X)$, since it doesn't matter which $x_0 \in X$ we pick.

Finally, recall that we originally defined "simply connected" as saying that any two paths with matching endpoints were homotopic. It's possible to weaken this condition and then rephrase it using fundamental groups.

Exercise 65.2.12. Let X be a path-connected space. Prove that X is **simply connected** if and only if $\pi_1(X)$ is the trivial group. (One direction is easy; the other is a little trickier.)

This is the "usual" definition of simply connected.

§65.3 Fundamental groups are invariant under homeomorphism

One quick shorthand I will introduce to clean up the discussion:

Definition 65.3.1. By $f\colon (X, x_0) \to (Y, y_0)$, we will mean that $f\colon X \to Y$ is a continuous function of spaces which also sends the point x_0 to y_0.

Let X and Y be topological spaces and $f\colon (X, x_0) \to (Y, y_0)$. We now want to relate the fundamental groups of X and Y.

Recall that a loop γ in (X, x_0) is a map $\gamma\colon [0,1] \to X$ with $\gamma(0) = \gamma(1) = x_0$. Then if we consider the composition

$$[0,1] \xrightarrow{\gamma} (X, x_0) \xrightarrow{f} (Y, y_0)$$

then we get straight-away a loop in Y at y_0! Let's call this loop $f_\sharp\gamma$.

Lemma 65.3.2 ($f_\sharp$ is homotopy invariant)
If $\gamma_1 \simeq \gamma_2$ are path-homotopic, then in fact

$$f_\sharp\gamma_1 \simeq f_\sharp\gamma_2.$$

Proof. Just take the homotopy h taking γ_1 to γ_2 and consider $f \circ h$. □

It's worth noting at this point that if X and Y are homeomorphic, then their fundamental groups are all isomorphic. Indeed, let $f\colon X \to Y$ and $g\colon Y \to X$ be mutually inverse continuous maps. Then one can check that $f_\sharp\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ and $g_\sharp\colon \pi_1(Y, y_0) \to \pi_1(X, x_0)$ are inverse maps between the groups (assuming $f(x_0) = y_0$ and $g(y_0) = x_0$).

§65.4 Higher homotopy groups

Why the notation π_1 for the fundamental group? And what are $\pi_2, \dots$? The answer lies in the following rephrasing:

> **Question 65.4.1.** Convince yourself that a loop is the same thing as a continuous function $S^1 \to X$.

It turns out we can define homotopy for things other than paths. Two functions $f, g\colon Y \to X$ are **homotopic** if there exists a continuous function $Y \times [0,1] \to X$ which continuously deforms f to g. So everything we did above was just the special case $Y = S^1$.

For general n, the group $\pi_n(X)$ is defined as the homotopy classes of the maps $S^n \to X$. The group operation is a little harder to specify. You have to show that S^n is homeomorphic to $[0,1]^n$ with some endpoints fused together; for example S^1 is $[0,1]$ with 0 fused to 1. Once you have these cubes, you can merge them together on a face. (Again, I'm being terribly imprecise, deliberately.)

For $n \neq 1$, π_n behaves somewhat differently than π_1. (You might not be surprised, as S^n is simply connected for all $n \geq 2$ but not when $n = 1$.) In particular, it turns out that $\pi_n(X)$ is an abelian group for all $n \geq 2$.

Let's see some examples.

Example 65.4.2 ($\pi_n(S^n) \cong \mathbb{Z}$)

As we saw, $\pi_1(S^1) \cong \mathbb{Z}$; given the base circle S^1, we can wrap a second circle around it as many times as we want. In general, it's true that $\pi_n(S^n) \cong \mathbb{Z}$.

Example 65.4.3 ($\pi_n(S^m) \cong \{1\}$ when $n < m$)

We saw that $\pi_1(S^2) \cong \{1\}$, because a circle in S^2 can just be reeled in to a point. It turns out that similarly, any smaller n-dimensional sphere can be reeled in on the surface of a bigger m-dimensional sphere. So in general, $\pi_n(S^m)$ is trivial for $n < m$.

However, beyond these observations, the groups behave quite weirdly. Here is a table of $\pi_n(S^m)$ for $1 \le m \le 8$ and $2 \le n \le 10$, so you can see what I'm talking about. (Taken from Wikipedia.)

$\pi_n(S^m)$	2	3	4	5	6	7	8	9	10
$m = 1$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/12\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/15\mathbb{Z}$
3		$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/12\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/15\mathbb{Z}$
4			$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$
5				$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/24\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
6					$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/24\mathbb{Z}$	$\{1\}$
7						$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/24\mathbb{Z}$
8							$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

Actually, it turns out that if you can compute $\pi_n(S^m)$ for every m and n, then you can essentially compute *any* homotopy classes. Thus, computing $\pi_n(S^m)$ is sort of a lost cause in general, and the mixture of chaos and pattern in the above table is a testament to this.

§65.5 Homotopy equivalent spaces

Prototypical example for this section: A disk is homotopy equivalent to a point, an annulus is homotopy equivalent to S^1.

Up to now I've abused notation and referred to "path homotopy" as just "homotopy" for two paths. I will unfortunately continue to do so (and so any time I say two paths are homotopic, you should assume I mean "path-homotopic"). But let me tell you what the general definition of homotopy is first.

Definition 65.5.1. Let $f, g \colon X \to Y$ be continuous functions. A **homotopy** is a continuous function $F \colon X \times [0,1] \to Y$, which we'll write $F_s(x)$ for $s \in [0,1]$, $x \in X$, such that

$$F_0(x) = f(x) \text{ and } F_1(x) = g(x) \text{ for all } x \in X.$$

If such a function exists, then f and g are **homotopic**.

Intuitively this is once again "deforming f to g". You might notice this is almost exactly the same definition as path-homotopy, except that f and g are any functions instead of paths, and hence there's no restriction on keeping some "endpoints" fixed through the deformation.

This homotopy can be quite dramatic:

Example 65.5.2

The zero function $z \mapsto 0$ and the identity function $z \mapsto z$ are homotopic as functions $\mathbb{C} \to \mathbb{C}$. The necessary deformation is

$$[0,1] \times \mathbb{C} \to \mathbb{C} \text{ by } (s,z) \mapsto sz.$$

I bring this up because I want to define:

Definition 65.5.3. Let X and Y be spaces. They are **homotopy equivalent** if there exist continuous functions $f\colon X \to Y$ and $g\colon Y \to X$ such that

(i) $f \circ g\colon Y \to Y$ is homotopic to the identity map on Y, and

(ii) $g \circ f\colon X \to X$ is homotopic to the identity map on X.

If a topological space is homotopy equivalent to a point, then it is said to be **contractible**.

Question 65.5.4. Why are two homeomorphic spaces also homotopy equivalent?

Intuitively, you can think of this as a more generous form of stretching and bending than homeomorphism: we are allowed to compress huge spaces into single points.

Example 65.5.5 ($\mathbb{C}$ is contractible)

Consider the topological spaces $\mathbb{C}$ and the space consisting of the single point $\{0\}$. We claim these spaces are homotopy equivalent (can you guess what f and g are?) Indeed, the two things to check are

(i) $\mathbb{C} \to \{0\} \hookrightarrow \mathbb{C}$ by $z \mapsto 0 \mapsto 0$ is homotopy equivalent to the identity on $\mathbb{C}$, which we just saw, and

(ii) $\{0\} \hookrightarrow \mathbb{C} \to \{0\}$ by $0 \mapsto 0 \mapsto 0$, which *is* the identity on $\{0\}$.

Here by $\hookrightarrow$ I just mean $\to$ in the special case that the function is just an "inclusion".

Remark 65.5.6 — $\mathbb{C}$ cannot be *homeomorphic* to a point because there is no bijection of sets between them.

Example 65.5.7 ($\mathbb{C} \setminus \{0\}$ is homotopy equivalent to S^1)

Consider the topological spaces $\mathbb{C} \setminus \{0\}$, the **punctured plane**, and the circle S^1 viewed as a subset of $\mathbb{C}$. We claim these spaces are actually homotopy equivalent! The necessary functions are the inclusion

$$S^1 \hookrightarrow \mathbb{C} \setminus \{0\}$$

and the function

$$\mathbb{C} \setminus \{0\} \to S^1 \quad \text{by} \quad z \mapsto \frac{z}{|z|}.$$

You can check that these satisfy the required condition.

Remark 65.5.8 — On the other hand, $\mathbb{C} \setminus \{0\}$ cannot be *homeomorphic* to S^1. One can make S^1 disconnected by deleting two points; the same is not true for $\mathbb{C} \setminus \{0\}$.

Example 65.5.9 (Disk = Point, Annulus = Circle)
By the same token, a disk is homotopic to a point; an annulus is homotopic to a circle. (This might be a little easier to visualize, since it's finite.)

I bring these up because it turns out that

Algebraic topology can't distinguish between homotopy equivalent spaces.

More precisely,

Theorem 65.5.10 (Homotopy equivalent spaces have isomorphic fundamental groups)
Let X and Y be path-connected, homotopy-equivalent spaces. Then $\pi_n(X) \cong \pi_n(Y)$ for every positive integer n.

Proof. Let $\gamma\colon [0,1]^n \to X$ be a S^n. Let $f\colon X \to Y$ and $g\colon Y \to X$ be maps witnessing that X and Y are homotopy equivalent (meaning $f \circ g$ and $g \circ f$ are each homotopic to the identity). Then the composition

$$[0,1]^n \xrightarrow{\gamma} X \xrightarrow{f} Y$$

is a S^n in Y and hence f induces a natural homomorphism $\pi_n(X) \to \pi_n(Y)$. Similarly g induces a natural homomorphism $\pi_n(Y) \to \pi_n(X)$. The conditions on f and g now say exactly that these two homomorphisms are inverse to each other, meaning the maps are isomorphisms. □

In particular,

Question 65.5.11. What are the fundamental groups of contractible spaces?

That means, for example, that algebraic topology can't tell the following homotopic subspaces of $\mathbb{R}^2$ apart.

♀ ♂

§65.6 The pointed homotopy category

This section is meant to be read by those who know some basic category theory. Those of you that don't should come back after reading Chapters 67 and 68. Those of you that do will enjoy how succinctly we can summarize the content of this chapter using categorical notions.

Definition 65.6.1. The **pointed homotopy category** hTop_* is defined as follows.

- Objects: **pointed spaces**; that is, a pair (X, x_0) of spaces X with a distinguished basepoint x_0, and
- Morphisms: *homotopy classes* of continuous functions $(X, x_0) \to (Y, y_0)$.

In particular, two path-connected spaces are isomorphic in this category exactly when they are homotopy equivalent. Then we can summarize many of the preceding results as follows:

Theorem 65.6.2 (Functorial interpretation of fundamental groups)

There is a functor

$$\pi_1 \colon \mathsf{hTop}_* \to \mathsf{Grp}$$

sending

$$\begin{array}{ccc} (X, x_0) & \dashrightarrow & \pi_1(X, x_0) \\ f \downarrow & & \downarrow f_\sharp \\ (Y, y_0) & \dashrightarrow & \pi_1(Y, y_0) \end{array}$$

The fact that π_1 is a functor instead of merely assigns some group $\pi_1(X, x_0)$ to each pointed topological space (X, x_0) automatically implies several nice things, like:

- The functor bundles the information of $f_\sharp$, including the fact that it respects composition. In the categorical language, $f_\sharp$ is $\pi_1(f)$.
- Homotopic spaces have isomorphic fundamental groups (since the spaces are isomorphic in hTop, and functors preserve isomorphism by Theorem 68.2.8). In fact, you'll notice that the proofs of Theorem 68.2.8 and Theorem 65.5.10 are secretly identical to each other.
- If maps $f, g \colon (X, x_0) \to (Y, y_0)$ are homotopic, then $f_\sharp = g_\sharp$. This is basically Lemma 65.3.2.

Remark 65.6.3 — In fact, $\pi_1(X, x_0)$ is the set of arrows $(S^1, 1) \to (X, x_0)$ in hTop_*, so this is actually a covariant Yoneda functor (Example 68.2.6), except with target Grp instead of Set.

§65.7 A few harder problems to think about

Problem 65A (Harmonic fan)**.** Exhibit a subspace X of the metric space $\mathbb{R}^2$ which is path-connected but for which a point p can be found such that any r-neighborhood of p with $r < 1$ is not path-connected.

Problem 65B† (Special case of Seifert-van Kampen)**.** Let X be a topological space. Suppose U and V are connected open subsets of X, with $X = U \cup V$, so that $U \cap V$ is nonempty and path-connected.

Prove that if $\pi_1(U) = \pi_1(V) = \{1\}$ then $\pi_1(X) = \{1\}$.

Remark 65.7.1 — The **Seifert–van Kampen theorem** generalizes this for $\pi_1(U)$ and $\pi_1(V)$ any groups; it gives a formula for calculating $\pi_1(X)$ in terms of $\pi_1(U)$,

> $\pi_1(V)$, $\pi_1(U \cap V)$. The proof is much the same.
> Unfortunately, this does not give us a way to calculate $\pi_1(S^1)$, because it is not possible to write $S^1 = U \cup V$ for $U \cap V$ *connected*.

Problem 65C (RMM 2013)**.** Let $n \geq 2$ be a positive integer. A stone is placed at each vertex of a regular $2n$-gon. A move consists of selecting an edge of the $2n$-gon and swapping the two stones at the endpoints of the edge. Prove that if a sequence of moves swaps every pair of stones exactly once, then there is some edge never used in any move.

(This last problem doesn't technically have anything to do with the chapter, but the "gut feeling" which motivates the solution is very similar.)

66 Covering projections

A few chapters ago we talked about what a fundamental group was, but we didn't actually show how to compute any of them except for the most trivial case of a simply connected space. In this chapter we'll introduce the notion of a *covering projection*, which will let us see how some of these groups can be found.

§66.1 Even coverings and covering projections

Prototypical example for this section: $\mathbb{R}$ *covers* S^1.

What we want now is a notion where a big space E, a "covering space", can be projected down onto a base space B in a nice way. Here is the notion of "nice":

Definition 66.1.1. Let $p\colon E \to B$ be a continuous function. Let U be an open set of B. We call U **evenly covered** (by p) if $p^{\text{pre}}(U)$ is a disjoint union of open sets of E (possibly infinite) such that p restricted to any of these sets is a homeomorphism.

Picture:

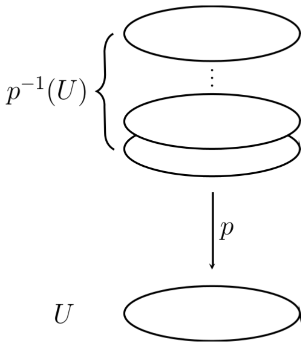


Image from [Wo]

All we're saying is that U is evenly covered if its pre-image is a bunch of copies of it. (Actually, a little more: each of the pancakes is homeomorphic to U, but we also require that p is the homeomorphism.)

Definition 66.1.2. A **covering projection** $p\colon E \to B$ is a surjective continuous map such that every base point $b \in B$ has an open neighborhood $U \ni b$ which is evenly covered by p.

Exercise 66.1.3 (On requiring surjectivity of p)**.** Let $p\colon E \to B$ be satisfying this definition, except that p need not be surjective. Show that the image of p is a disjoint union of connected components of B. Thus if B is connected and E is nonempty, then $p\colon E \to B$ is already surjective. For this reason, some authors omit the surjectivity hypothesis as usually B is path-connected.

Here is the most stupid example of a covering projection.

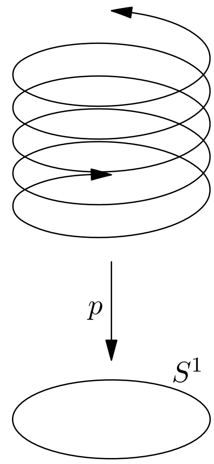
Example 66.1.4 (Tautological covering projection)

Let's take n disconnected copies of any space B: formally, $E = B \times \{1, \dots, n\}$

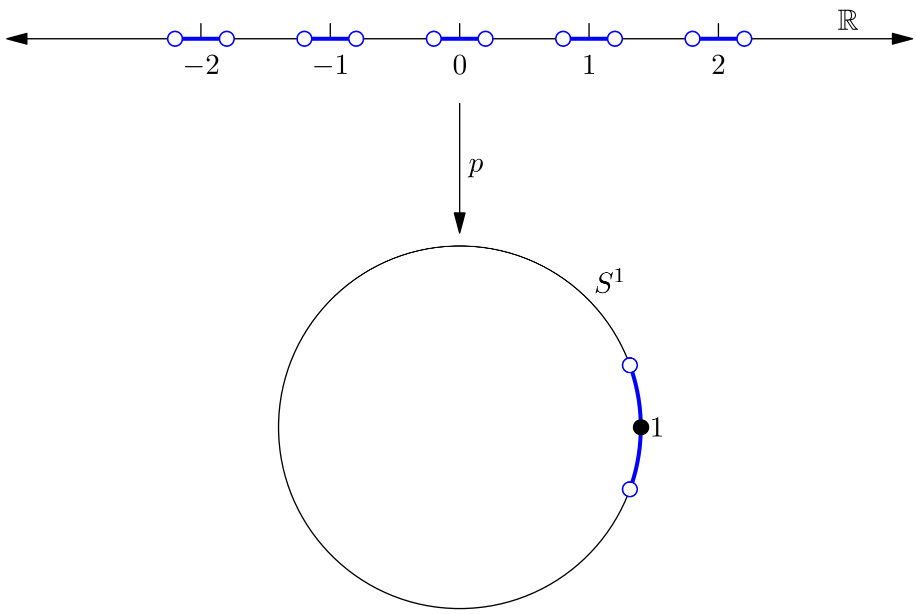
with the discrete topology on $\{1, \dots, n\}$. Then there exists a tautological covering projection $E \to B$ by $(x, m) \mapsto x$; we just project all n copies.
This is a covering projection because *every* open set in B is evenly covered.

This is not really that interesting because $B \times [n]$ is not path-connected.
A much more interesting example is that of $\mathbb{R}$ and S^1.

Example 66.1.5 (Covering projection of S^1)
Take $p \colon \mathbb{R} \to S^1$ by $\theta \mapsto e^{2\pi i \theta}$. This is essentially wrapping the real line into a single helix and projecting it down.



We claim this is a covering projection. Indeed, consider the point $1 \in S^1$ (where we view S^1 as the unit circle in the complex plane). We can draw a small open neighborhood of it whose pre-image is a bunch of copies in $\mathbb{R}$.



Note that not all open neighborhoods work this time: notably, $U = S^1$ does not work because the pre-image would be the entire $\mathbb{R}$ which is not homeomorphic with S^1.

Example 66.1.6 (Covering of S^1 by itself)

The map $S^1 \to S^1$ by $z \mapsto z^3$ is also a covering projection. Can you see why?

Example 66.1.7 (Covering projections of $\mathbb{C} \setminus \{0\}$)

For those comfortable with complex arithmetic,

(a) The exponential map $\exp\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is a covering projection.

(b) For each n, the nth power map $-^n : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is a covering projection.

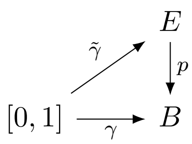
§66.2 Lifting theorem

Prototypical example for this section: $\mathbb{R}$ covers S^1.

Now here's the key idea: we are going to try to interpret loops in B as paths in $\mathbb{R}$. This is often much simpler. For example, we had no idea how to compute the fundamental group of S^1, but the fundamental group of $\mathbb{R}$ is just the trivial group. So if we can interpret loops in S^1 as paths in $\mathbb{R}$, that might (and indeed it does!) make computing $\pi_1(S^1)$ tractable.

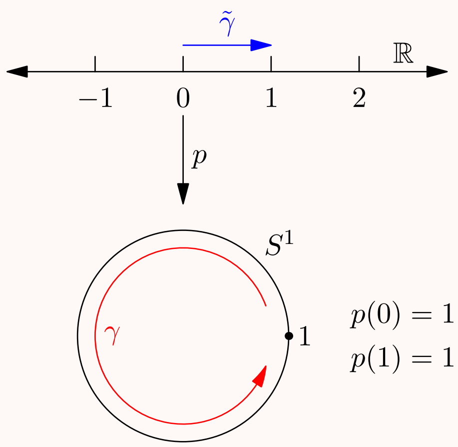
Definition 66.2.1. Let $\gamma\colon [0,1] \to B$ be a path and $p\colon E \to B$ a covering projection. A **lifting** of γ is a path $\tilde{\gamma}\colon [0,1] \to E$ such that $p \circ \tilde{\gamma} = \gamma$.

Picture:



Example 66.2.2 (Typical example of lifting)

Take $p\colon \mathbb{R} \to S^1 \subseteq \mathbb{C}$ by $\theta \mapsto e^{2\pi i\theta}$ (so S^1 is considered again as the unit circle). Consider the path γ in S^1 which starts at $1 \in \mathbb{C}$ and wraps around S^1 once, counterclockwise, ending at 1 again. In symbols, $\gamma\colon [0,1] \to S^1$ by $t \mapsto e^{2\pi it}$. Then one lifting $\tilde{\gamma}$ is the path which walks from 0 to 1. In fact, *for any integer n*, walking from n to $n+1$ works.



Similarly, the counterclockwise path from $1 \in S^1$ to $-1 \in S^1$ has a lifting: for some integer n, the path from n to $n + \frac{1}{2}$.

The above is the primary example of a lifting. It seems like we have the following structure: given a path γ in B starting at b_0, we start at any point in the fiber $p^{\text{pre}}(b_0)$. (In our prototypical example, $B = S^1$, $b_0 = 1 \in \mathbb{C}$ and that's why we start at any integer n.) After that we just trace along the path in B, and we get a corresponding path in E.

Question 66.2.3. Take a path γ in S^1 with $\gamma(0) = 1 \in \mathbb{C}$. Convince yourself that once we select an integer $n \in \mathbb{R}$, then there is exactly one lifting starting at n.

It turns out this is true more generally.

Theorem 66.2.4 (Lifting paths)

Suppose $\gamma\colon [0,1] \to B$ is a path with $\gamma(0) = b_0$, and $p\colon (E, e_0) \to (B, b_0)$ is a covering projection. Then there exists a *unique* lifting $\tilde{\gamma}\colon [0,1] \to E$ such that $\tilde{\gamma}(0) = e_0$.

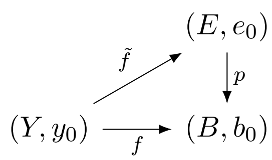
Proof. For every point $b \in B$, consider an evenly covered open neighborhood U_b in B. Then the family of open sets

$$\{\gamma^{\text{pre}}(U_b) \mid b \in B\}$$

is an open cover of $[0,1]$. As $[0,1]$ is compact we can take a finite subcover. Thus we can chop $[0,1]$ into finitely many interior-disjoint closed intervals $[0,1] = I_1 \sqcup I_2 \sqcup \dots \sqcup I_N$ in that order, such that for every I_k, $\gamma^{\text{img}}(I_k)$ is contained in some U_b.

We'll construct $\tilde{\gamma}$ interval by interval now, starting at I_1. Initially, place a robot at $e_0 \in E$ and a mouse at $b_0 \in B$. For each interval I_k, the mouse moves around according to however γ behaves on I_k. But the whole time it's in some evenly covered U_k; the fact that p is a covering projection tells us that there are several copies of U_k living in E. Exactly one of them, say V_k, contains our robot. So the robot just mimics the mouse until it gets to the end of I_k. Then the mouse is in some new evenly covered U_{k+1}, and we can repeat. $\square$

The theorem can be generalized to a diagram



where Y is some general path-connected space, as follows.

Theorem 66.2.5 (General lifting criterion)

Let $f \colon (Y, y_0) \to (B, b_0)$ be continuous and consider a covering projection $p \colon (E, e_0) \to (B, b_0)$. (As usual, Y, B, E are path-connected.) Then a lifting $\tilde{f}$ with $\tilde{f}(y_0) = e_0$ exists if and only if

$$f_\sharp^{\text{img}}(\pi_1(Y, y_0)) \subseteq p_\sharp^{\text{img}}(\pi_1(E, e_0)),$$

i.e. the image of $\pi_1(Y, y_0)$ under f is contained in the image of $\pi_1(E, e_0)$ under p (both viewed as subgroups of $\pi_1(B, b_0)$). If this lifting exists, it is unique.

As $p_\sharp$ is injective, we actually have $p_\sharp^{\text{img}}(\pi_1(E, e_0)) \cong \pi_1(E, e_0)$. But in this case we are interested in the actual elements, not just the isomorphism classes of the groups.

Question 66.2.6. What happens if we put $Y = [0, 1]$?

Remark 66.2.7 (Lifting homotopies) — Here's another cool special case: Recall that a homotopy can be encoded as a continuous function $[0, 1] \times [0, 1] \to X$. But $[0, 1] \times [0, 1]$ is also simply connected. Hence given a homotopy $\gamma_1 \simeq \gamma_2$ in the base space B, we can lift it to get a homotopy $\tilde{\gamma}_1 \simeq \tilde{\gamma}_2$ in E.

Another nice application of this result is Chapter 33.

§66.3 Lifting correspondence

Prototypical example for this section: $(\mathbb{R}, 0)$ *covers* $(S^1, 1)$.

Let's return to the task of computing fundamental groups. Consider a covering projection $p \colon (E, e_0) \to (B, b_0)$.

A loop γ in B can be lifted uniquely to $\tilde{\gamma}$ in E which starts at e_0 and ends at some point e in the fiber $p^{\text{pre}}(b_0)$. You can easily check that this $e \in E$ does not change if we pick a different path γ' homotopic to γ.

Question 66.3.1. Look at the picture in Example 66.2.2.
Put one finger at $1 \in S^1$, and one finger on $0 \in \mathbb{R}$. Trace a loop homotopic to γ in S^1 (meaning, you can go backwards and forwards but you must end with exactly one full counterclockwise rotation) and follow along with the other finger in $\mathbb{R}$.
Convince yourself that you have to end at the point $1 \in \mathbb{R}$.

Thus every homotopy class of a loop at b_0 (i.e. an element of $\pi_1(B, b_0)$) can be associated with some e in the fiber of b_0. The below proposition summarizes this and more.

Proposition 66.3.2

Let $p\colon (E, e_0) \to (B, b_0)$ be a covering projection. Then we have a function of sets

$$\Phi\colon \pi_1(B, b_0) \to p^{\text{pre}}(b_0)$$

by $[\gamma] \mapsto \tilde{\gamma}(1)$, where $\tilde{\gamma}$ is the unique lifting starting at e_0. Furthermore,

- If E is path-connected, then Φ is surjective.
- If E is simply connected, then Φ is injective.

Question 66.3.3. Prove that E path-connected implies Φ is surjective. (This is really offensively easy.)

Proof. To prove the proposition, we've done everything except show that E simply connected implies Φ injective. To do this suppose that γ_1 and γ_2 are loops such that $\Phi([\gamma_1]) = \Phi([\gamma_2])$.

Applying lifting, we get paths $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ both starting at some point $e_0 \in E$ and ending at some point $e_1 \in E$. Since E is simply connected that means they are *homotopic*, and we can write a homotopy $F\colon [0,1] \times [0,1] \to E$ which unites them. But then consider the composition of maps

$$[0,1] \times [0,1] \xrightarrow{F} E \xrightarrow{p} B.$$

You can check $p \circ F$ is a homotopy from γ_1 to γ_2. Hence $[\gamma_1] = [\gamma_2]$, done. □

This motivates:

Definition 66.3.4. A **universal cover** of a space B is a covering projection $p\colon E \to B$ where E is simply connected (and in particular path-connected).

Abuse of Notation 66.3.5. When p is understood, we sometimes just say E is the universal cover of B.

Example 66.3.6 (Fundamental group of S^1)

Let's return to our standard $p\colon \mathbb{R} \to S^1$. Since $\mathbb{R}$ is simply connected, this is a universal cover of S^1. And indeed, the fiber of any point in S^1 is a copy of the integers: naturally in bijection with loops in S^1.

You can show (and it's intuitively obvious) that the bijection

$$\Phi\colon \pi_1(S^1) \leftrightarrow \mathbb{Z}$$

is in fact a group homomorphism if we equip $\mathbb{Z}$ with its additive group structure $\mathbb{Z}$. Since it's a bijection, this leads us to conclude $\pi_1(S^1) \cong \mathbb{Z}$.

§66.4 Regular coverings

Prototypical example for this section: $\mathbb{R} \to S^1$ *comes from* $n \cdot x = n + x$

Here's another way to generate some coverings. Let X be a topological space and G a group acting on its points. Thus for every g, we get a map $X \to X$ by

$$x \mapsto g \cdot x.$$

We require that this map is continuous[1] for every $g \in G$, and that the stabilizer of each point in X is trivial. Then we can consider a quotient space X/G defined by fusing any points in the same orbit of this action. Thus the points of X/G are identified with the orbits of the action. Then we get a natural "projection"

$$X \to X/G$$

by simply sending every point to the orbit it lives in.

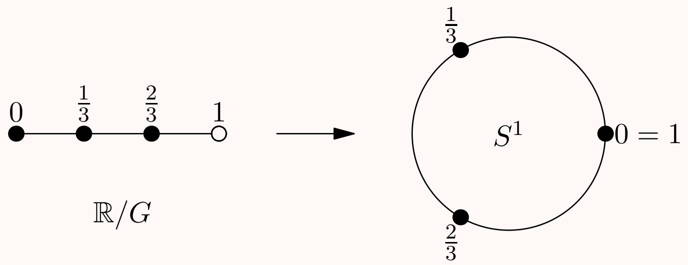
Definition 66.4.1. Such a projection is called **regular**. (Terrible, I know.)

Example 66.4.2 ($\mathbb{R} \to S^1$ is regular)

Let $G = \mathbb{Z}$, $X = \mathbb{R}$ and define the group action of G on X by

$$n \cdot x = n + x$$

You can then think of X/G as "real numbers modulo 1", with $[0, 1)$ a complete set of representatives and $0 \sim 1$.



So we can identify X/G with S^1 and the associated regular projection is just our usual $\exp\colon \theta \mapsto e^{2i\pi\theta}$.

Example 66.4.3 (The torus)

Let $G = \mathbb{Z} \times \mathbb{Z}$ and $X = \mathbb{R}^2$, and define the group action of G on X by $(m, n) \cdot (x, y) = (m + x, n + y)$. As $[0, 1)^2$ is a complete set of representatives, you can think of it as a unit square with the edges identified. We obtain the torus $S^1 \times S^1$ and a covering projection $\mathbb{R}^2 \to S^1 \times S^1$.

Example 66.4.4 ($\mathbb{RP}^2$)

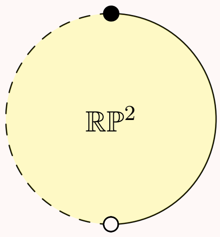
Let $G = \mathbb{Z}/2\mathbb{Z} = \langle T \mid T^2 = 1\rangle$ and let $X = S^2$ be the surface of the sphere, viewed as a subset of $\mathbb{R}^3$. We'll let G act on X by sending $T \cdot \vec{x} = -\vec{x}$; hence the orbits are pairs of opposite points (e.g. North and South pole).

Let's draw a picture of a space. All the orbits have size two: every point below the equator gets fused with a point above the equator. As for the points on the equator, we can take half of them; the other half gets fused with the corresponding antipodes.

Now if we flatten everything, you can think of the result as a disk with half its

[1]Another way of phrasing this: the action, interpreted as a map $G \times X \to X$, should be continuous, where G on the left-hand side is interpreted as a set with the discrete topology.

boundary: this is $\mathbb{RP}^2$ from before. The resulting space has a name: *real projective 2-space*, denoted $\mathbb{RP}^2$.



This gives us a covering projection $S^2 \to \mathbb{RP}^2$ (note that the pre-image of a sufficiently small patch is just two copies of it on S^2.)

Example 66.4.5 (Fundamental group of $\mathbb{RP}^2$)

As above, we saw that there was a covering projection $S^2 \to \mathbb{RP}^2$. Moreover the fiber of any point has size two. Since S^2 is simply connected, we have a natural bijection $\pi_1(\mathbb{RP}^2)$ to a set of size two; that is,

$$\left|\pi_1(\mathbb{RP}^2)\right| = 2.$$

This can only occur if $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}$, as there is only one group of order two!

Question 66.4.6. Show each of the continuous maps $x \mapsto g \cdot x$ is in fact a homeomorphism. (Name its continuous inverse).

§66.5 The algebra of fundamental groups

Prototypical example for this section: S^1, with fundamental group $\mathbb{Z}$.

Next up, we're going to turn functions between spaces into homomorphisms of fundamental groups.

Let X and Y be topological spaces and $f\colon (X, x_0) \to (Y, y_0)$. Recall that we defined a group homomorphism

$$f_\sharp \colon \pi_1(X, x_0) \to \pi_1(Y_0, y_0) \quad \text{by} \quad [\gamma] \mapsto [f \circ \gamma].$$

More importantly, we have:

Proposition 66.5.1

Let $p\colon (E, e_0) \to (B, b_0)$ be a covering projection of path-connected spaces. Then the homomorphism $p_\sharp \colon \pi_1(E, e_0) \to \pi_1(B, b_0)$ is *injective*. Hence $p_\sharp^{\text{img}}(\pi_1(E, e_0))$ is an isomorphic copy of $\pi_1(E, e_0)$ as a subgroup of $\pi_1(B, b_0)$.

Proof. We'll show $\ker p_\sharp$ is trivial. It suffices to show if γ is a nulhomotopic loop in B then its lift is nulhomotopic.

By definition, there's a homotopy $F\colon [0,1] \times [0,1] \to B$ taking γ to the constant loop 1_B. We can lift it to a homotopy $\tilde{F}\colon [0,1] \times [0,1] \to E$ that establishes $\tilde{\gamma} \simeq \tilde{1}_B$. But 1_E

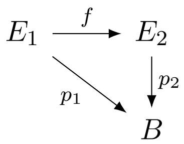
is a lift of 1_B (duh) and lifts are unique. □

Example 66.5.2 (Subgroups of $\mathbb{Z}$)
Let's look at the space S^1 with fundamental group $\mathbb{Z}$. The group $\mathbb{Z}$ has two types of subgroups:

- The trivial subgroup. This corresponds to the canonical projection $\mathbb{R} \to S^1$, since $\pi_1(\mathbb{R})$ is the trivial group ($\mathbb{R}$ is simply connected) and hence its image in $\mathbb{Z}$ is the trivial group.
- $n\mathbb{Z}$ for $n \geq 1$. This is given by the covering projection $S^1 \to S^1$ by $z \mapsto z^n$. The image of a loop in the covering S^1 is a "multiple of n" in the base S^1.

It turns out that these are the *only* covering projections of S^1 by path-connected spaces: there's one for each subgroup of $\mathbb{Z}$. (We don't care about disconnected spaces because, again, a covering projection via disconnected spaces is just a bunch of unrelated "good" coverings.) For this statement to make sense I need to tell you what it means for two covering projections to be equivalent.

Definition 66.5.3. Fix a space B. Given two covering projections $p_1 \colon E_1 \to B$ and $p_2 \colon E_2 \to B$ a **map of covering projections** is a continuous function $f \colon E_1 \to E_2$ such that $p_2 \circ f = p_1$.



Then two covering projections p_1 and p_2 are isomorphic if there are $f \colon E_1 \to E_2$ and $g \colon E_2 \to E_1$ such that $f \circ g = \mathrm{id}_{E_1}$ and $g \circ f = \mathrm{id}_{E_2}$.

Remark 66.5.4 (For category theorists) — The set of covering projections forms a category in this way.

It's an absolute miracle that this is true more generally: the greatest triumph of covering spaces is the following result. Suppose a space X satisfies some nice conditions, like:

Definition 66.5.5. A space X is called **locally connected** if for each point $x \in X$ and open neighborhood V of it, there is a connected open set U with $x \in U \subseteq V$.

Definition 66.5.6. A space X is **semi-locally simply connected** if for every point $x \in X$ there is an open neighborhood U such that all loops in U are nulhomotopic. (But the contraction need not take place in U.)

Example 66.5.7 (These conditions are weak)
Pretty much every space I've shown you has these two properties. In other words, they are rather mild conditions, and you can think of them as just saying "the space is not too pathological".

Then we get:

Theorem 66.5.8 (Group theory via covering spaces)

Suppose B is a locally connected, semi-locally simply connected space. Then:

- Every subgroup $H \subseteq \pi_1(B)$ corresponds to exactly one covering projection $p \colon E \to B$ with E path-connected (up to isomorphism).

  (Specifically, H is the image of $\pi_1(E)$ in $\pi_1(B)$ through $p_\sharp$.)

- Moreover, the *normal* subgroups of $\pi_1(B)$ correspond exactly to the regular covering projections.

Hence it's possible to understand the group theory of $\pi_1(B)$ completely in terms of the covering projections.

Moreover, this is how the "universal cover" gets its name: it is the one corresponding to the trivial subgroup of $\pi_1(B)$. Actually, you can show that it really is universal in the sense that if $p \colon E \to B$ is another covering projection, then E is in turn covered by the universal space. More generally, if $H_1 \subseteq H_2 \subseteq G$ are subgroups, then the space corresponding to H_2 can be covered by the space corresponding to H_1.

§66.6 A few harder problems to think about

problems