IX

Complex Analysis

Part IX: Contents

31 Holomorphic functions

Throughout this chapter, we denote by U an open subset of the complex plane, and by Ω an open subset which is also simply connected. The main references for this chapter were [Ya12; Ba10].

§31.1 The nicest functions on earth

In high school you were told how to differentiate and integrate real-valued functions. In this chapter on complex analysis, we'll extend it to differentiation and integration of complex-valued functions.

Big deal, you say. Calculus was boring enough. Why do I care about complex calculus?

Perhaps it's easiest to motivate things if I compare real analysis to complex analysis. In real analysis, your input lives inside the real line $\mathbb{R}$. This line is not terribly discerning – you can construct a lot of unfortunate functions. Here are some examples.

Example 31.1.1 (Optional: evil real functions)

You can skim over these very quickly: they're only here to make a point.

(a) The **Devil's Staircase** (or Cantor function) is a continuous function $H\colon [0,1] \to [0,1]$ which has derivative zero "almost everywhere", yet $H(0) = 0$ and $H(1) = 1$.

(b) The **Weierstraß function**

$$x \mapsto \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \cos\left(2015^n \pi x\right)$$

is continuous *everywhere* but differentiable *nowhere*.

(c) The function

$$x \mapsto \begin{cases} x^{100} & x \ge 0 \\ -x^{100} & x < 0 \end{cases}$$

has the first 99 derivatives but not the 100th one.

(d) If a function has all derivatives (we call these **smooth** functions), then it has a Taylor series. But for real functions that Taylor series might still be wrong. The function

$$x \mapsto \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

has derivatives at every point. But if you expand the Taylor series at $x = 0$, you get $0 + 0x + 0x^2 + \cdots$, which is wrong for *any* $x > 0$ (even $x = 0.0001$).

Let's even put aside the pathology. If I tell you the value of a real smooth function on the interval $[-1, 1]$, that still doesn't tell you anything about the function as a whole. It could be literally anything, because it's somehow possible to "fuse together" smooth functions.

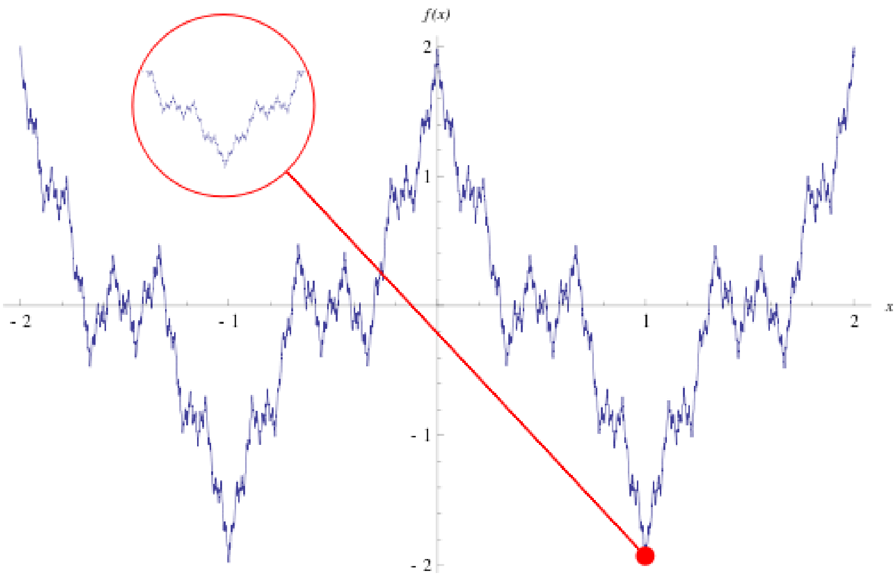


Figure 31.1: The Weierstraß Function (image from [**Ee**]).

So what about complex functions? If you consider them as functions $\mathbb{R}^2 \to \mathbb{R}^2$, you now have the interesting property that you can integrate along things that are not line segments: you can write integrals across curves in the plane. But $\mathbb{C}$ has something more: it is a *field*, so you can *multiply* and *divide* two complex numbers.

So we restrict our attention to differentiable functions called *holomorphic functions*. It turns out that the multiplication on $\mathbb{C}$ makes all the difference. The primary theme in what follows is that holomorphic functions are *really, really nice*, and that knowing tiny amounts of data about the function can determine all its values.

The two main highlights of this chapter, from which all other results are more or less corollaries:

- Contour integrals of loops are always zero.
- A holomorphic function is essentially given by its Taylor series; in particular, single-differentiable implies infinitely differentiable. Thus, holomorphic functions behave quite like polynomials.

Some of the resulting corollaries:

- It'll turn out that knowing the values of a holomorphic function on the boundary of the unit circle will tell you the values in its interior.
- Knowing the values of the function at $1, \frac{1}{2}, \frac{1}{3}, \dots$ are enough to determine the whole function!
- Bounded holomorphic functions $\mathbb{C} \to \mathbb{C}$ must be constant.
- And more...

As [**Pu02**] writes: "Complex analysis is the good twin and real analysis is the evil one: beautiful formulas and elegant theorems seem to blossom spontaneously in the complex domain, while toil and pathology rule the reals".

§31.2 Complex differentiation

Prototypical example for this section: Polynomials are holomorphic; $\overline{z}$ is not.

Let $f \colon U \to \mathbb{C}$ be a complex function. Then for some $z_0 \in U$, we define the **derivative** at z_0 to be

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

Note that this limit may not exist; when it does we say f is **differentiable** at z_0.

What do I mean by a "complex" limit $h \to 0$? It's what you might expect: for every $\varepsilon > 0$ there should be a $\delta > 0$ such that

$$0 < |h| < \delta \implies \left| \frac{f(z_0 + h) - f(z_0)}{h} - L \right| < \varepsilon.$$

If you like topology, you are encouraged to think of this in terms of open neighborhoods in the complex plane. (This is why we require U to be open: it makes it possible to take δ-neighborhoods in it.)

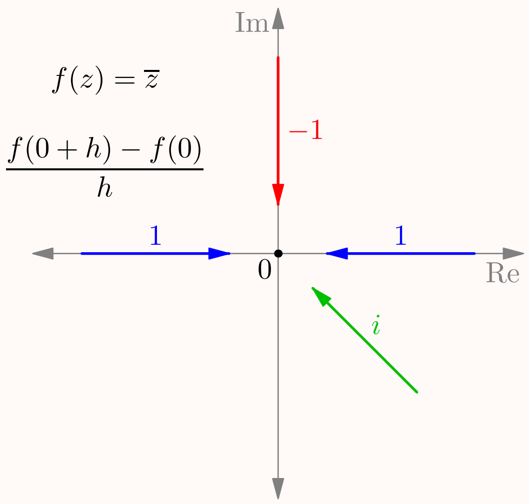
But note that having a complex derivative is actually much stronger than a real function having a derivative. In the real line, h can only approach zero from below and above, and for the limit to exist we need the "left limit" to equal the "right limit". But the complex numbers form a *plane*: h can approach zero from many directions, and we need all the limits to be equal.

Example 31.2.1 (Important: conjugation is *not* holomorphic)

Let $f(z) = \overline{z}$ be complex conjugation, $f \colon \mathbb{C} \to \mathbb{C}$. This function, despite its simple nature, is not holomorphic! Indeed, at $z = 0$ we have,

$$\frac{f(h) - f(0)}{h} = \frac{\overline{h}}{h}.$$

This does not have a limit as $h \to 0$, because depending on "which direction" we approach zero from we have different values.



If a function $f \colon U \to \mathbb{C}$ is complex differentiable at all the points in its domain it is called **holomorphic**. In the special case of a holomorphic function with domain $U = \mathbb{C}$, we call the function **entire**.[1]

[1] Sorry, I know the word "holomorphic" sounds so much cooler. I'll try to do things more generally for that sole reason.

Example 31.2.2 (Examples of holomorphic functions)

In all the examples below, the derivative of the function is the same as in their real analogues (e.g. the derivative of e^z is e^z).

(a) Any polynomial $z \mapsto z^n + c_{n-1}z^{n-1} + \cdots + c_0$ is holomorphic.

(b) The complex exponential $\exp\colon x + yi \mapsto e^x(\cos y + i \sin y)$ can be shown to be holomorphic.

(c) sin and cos are holomorphic when extended to the complex plane by $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ and $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$.

(d) As usual, the sum, product, chain rules and so on apply, and hence **sums, products, nonzero quotients, and compositions of holomorphic functions are also holomorphic**.

You are welcome to try and prove these results, but I won't bother to do so.

§31.3 Contour integrals

Prototypical example for this section: $\oint_\gamma z^m \, dz$ *around the unit circle.*

In the real line we knew how to integrate a function across a line segment $[a, b]$: essentially, we'd "follow along" the line segment adding up the values of f we see to get some area. Unlike in the real line, in the complex plane we have the power to integrate over arbitrary paths: for example, we might compute an integral around a unit circle. A contour integral lets us formalize this.

First of all, if $f\colon \mathbb{R} \to \mathbb{C}$ and $f(t) = u(t) + iv(t)$ for $u, v \in \mathbb{R}$, we can define an integral $\int_a^b$ by just adding the real and imaginary parts:

$$\int_a^b f(t) \, dt = \left(\int_a^b u(t) \, dt \right) + i \left(\int_a^b v(t) \, dt \right).$$

Now let $\alpha\colon [a, b] \to \mathbb{C}$ be a path, thought of as a complex differentiable[2] function. Such a path is called a **contour**, and we define its **contour integral** by

$$\oint_\alpha f(z) \, dz = \int_a^b f(\alpha(t)) \cdot \alpha'(t) \, dt.$$

You can almost think of this as a u-substitution (which is where the α' comes from). In particular, it turns out this integral does not depend on how α is "parametrized": a circle given by

$$[0, 2\pi] \to \mathbb{C}\colon t \mapsto e^{it}$$

and another circle given by

$$[0, 1] \to \mathbb{C}\colon t \mapsto e^{2\pi it}$$

and yet another circle given by

$$[0, 1] \to \mathbb{C}\colon t \mapsto e^{2\pi it^5}$$

[2]This isn't entirely correct here: you want the path α to be continuous and mostly differentiable, but you allow a finite number of points to have "sharp bends"; in other words, you can consider paths which are combinations of n smooth pieces. But for this we also require that α has "bounded length".

will all give the same contour integral, because the paths they represent have the same geometric description: "run around the unit circle once".

In what follows I try to use α for general contours and γ in the special case of loops.

Let's see an example of a contour integral.

Theorem 31.3.1

Take $\gamma\colon [0, 2\pi] \to \mathbb{C}$ to be the unit circle specified by

$$t \mapsto e^{it}.$$

Then for any integer m, we have

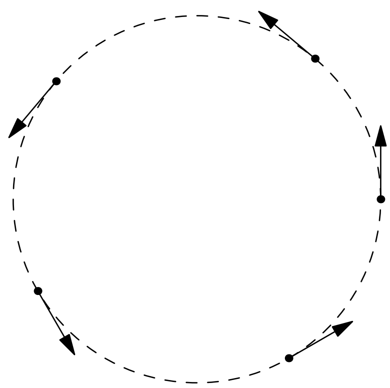
$$\oint_\gamma z^m \, dz = \begin{cases} 2\pi i & m = -1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. The derivative of e^{it} is ie^{it}. So, by definition the answer is the value of

$$\begin{aligned} \int_0^{2\pi} (e^{it})^m \cdot (ie^{it}) \, dt &= \int_0^{2\pi} i(e^{it})^{1+m} \, dt \\ &= i \int_0^{2\pi} \cos[(1+m)t] + i \sin[(1+m)t] \, dt \\ &= -\int_0^{2\pi} \sin[(1+m)t] \, dt + i \int_0^{2\pi} \cos[(1+m)t] \, dt. \end{aligned}$$

This is now an elementary calculus question. One can see that this equals $2\pi i$ if $m = -1$ and otherwise the integrals vanish. $\square$

Let me try to explain why this intuitively ought to be true for $m = 0$. In that case we have $\oint_\gamma 1 \, dz$. So as the integral walks around the unit circle, it "sums up" all the tangent vectors at every point (that's the direction it's walking in), multiplied by 1. And given the nice symmetry of the circle, it should come as no surprise that everything cancels out. The theorem says that even if we multiply by z^m for $m \neq -1$, we get the same cancellation.



Definition 31.3.2. Given $\alpha\colon [0, 1] \to \mathbb{C}$, we denote by $\overline{\alpha}$ the "backwards" contour $\overline{\alpha}(t) = \alpha(1 - t)$.

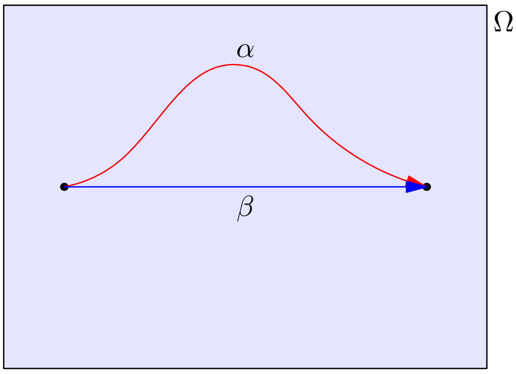
Question 31.3.3. What's the relation between $\oint_\alpha f\, dz$ and $\oint_{\overline{\alpha}} f\, dz$? Prove it.

This might seem a little boring. Things will get really cool really soon, I promise.

§31.4 Cauchy-Goursat theorem

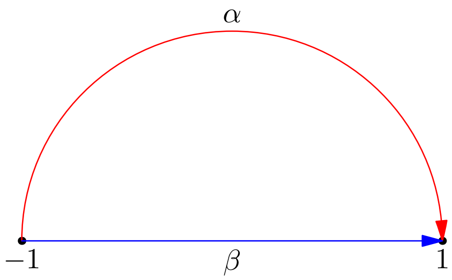
Prototypical example for this section: $\oint_\gamma z^m\, dz = 0$ *for* $m \ge 0$. *But if* $m < 0$, *Cauchy's theorem does not apply.*

Let $\Omega \subseteq \mathbb{C}$ be simply connected (for example, $\Omega = \mathbb{C}$), and consider two paths α, β with the same start and end points.



What's the relation between $\oint_\alpha f(z)\, dz$ and $\oint_\beta f(z)\, dz$? You might expect there to be some relation between them, considering that the space Ω is simply connected. But you probably wouldn't expect there to be *much* of a relation.

As a concrete example, let $\Psi \colon \mathbb{C} \to \mathbb{C}$ be the function $z \mapsto z - \operatorname{Re}[z]$ (for example, $\Psi(2015 + 3i) = 3i$). Let's consider two paths from -1 to 1. Thus β is walking along the real axis, and α which follows an upper semicircle.



Obviously $\oint_\beta \Psi(z)\, dz = 0$. But heaven knows what $\oint_\alpha \Psi(z)\, dz$ is supposed to equal. We can compute it now just out of non-laziness. If you like, you are welcome to compute it yourself (it's a little annoying but not hard). If I myself didn't mess up, it is

$$\oint_\alpha \Psi(z)\, dz = -\oint_{\overline{\alpha}} \Psi(z)\, dz = -\int_0^\pi (i\sin(t)) \cdot ie^{it}\, dt = \frac{1}{2}\pi i$$

which in particular is not zero.

But somehow Ψ is not a really natural function. It's not respecting any of the nice, multiplicative structure of $\mathbb{C}$ since it just rudely lops off the real part of its inputs. More precisely,

Question 31.4.1. Show that $\Psi(z) = z - \operatorname{Re}[z]$ is not holomorphic. (Hint: $\overline{z}$ is not holomorphic.)

Now here's a miracle: for holomorphic functions, the two integrals are *always equal*. Equivalently, (by considering α followed by $\overline{\beta}$) contour integrals of loops are always zero. This is the celebrated Cauchy-Goursat theorem (also called the Cauchy integral theorem, but later we'll have a "Cauchy Integral Formula" so blah).

Theorem 31.4.2 (Cauchy-Goursat theorem)

Let γ be a loop, and $f\colon \Omega \to \mathbb{C}$ a holomorphic function where Ω is open in $\mathbb{C}$ and simply connected. Then

$$\oint_\gamma f(z)\, dz = 0.$$

Remark 31.4.3 (Sanity check) — This might look surprising considering that we saw $\oint_\gamma z^{-1}\, dz = 2\pi i$ earlier. The subtlety is that z^{-1} is not even defined at $z = 0$. On the other hand, the function $\mathbb{C} \setminus \{0\} \to \mathbb{C}$ by $z \mapsto \frac{1}{z}$ *is* holomorphic! The defect now is that $\Omega = \mathbb{C} \setminus \{0\}$ is not simply connected. So the theorem passes our sanity checks, albeit barely.

The typical proof of Cauchy's Theorem assumes additionally that the partial derivatives of f are continuous and then applies the so-called Green's theorem. But it was Goursat who successfully proved the fully general theorem we've stated above, which assumed only that f was holomorphic.

Anyways, the theorem implies that $\oint_\gamma z^m\, dz = 0$ when $m \ge 0$. So much for all our hard work earlier. But so far we've only played with circles. This theorem holds for *any* contour which is a loop. So what else can we do?

§31.5 Cauchy's integral theorem

We now present a stunning application of Cauchy-Goursat, a "representation theorem": essentially, it says that values of f inside a disk are determined by just the values on the boundary! In fact, we even write down the exact formula. As [Ya12] says, "any time a certain type of function satisfies some sort of representation theorem, it is likely that many more deep theorems will follow." Let's pull back the curtain:

Theorem 31.5.1 (Cauchy's integral formula)

Let $\gamma\colon [0, 2\pi] \to \mathbb{C}$ be a circle in the plane given by $t \mapsto Re^{it}$, which bounds a disk D. Suppose $f\colon U \to \mathbb{C}$ is holomorphic such that U contains the circle and its interior. Then for any point a in the interior of D, we have

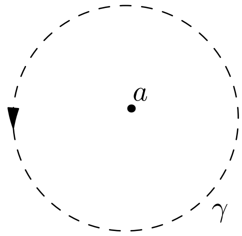
$$f(a) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-a}\, dz.$$

Note that we don't require U to be simply connected, but the reason is pretty silly: we're only going to ever integrate f over D, which is an open disk, and hence the disk is simply connected anyways.

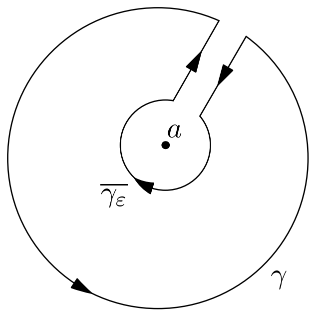
The presence of $2\pi i$, which you saw earlier in the form $\oint_{\text{circle}} z^{-1}\,dz$, is no accident. In fact, that's the central result we're going to use to prove the result.

Remark 31.5.2 — With the introduction of meromorphic functions next chapter, we will see how to intuitively derive this formula in Remark 32.3.5.

Proof. There are several proofs out there, but I want to give the one that really draws out the power of Cauchy's theorem. Here's the picture we have: there's a point a sitting inside a circle γ, and we want to get our hands on the value $f(a)$.



We're going to do a trick: construct a **keyhole contour** $\Gamma_{\delta,\varepsilon}$ which has an outer circle γ, plus an inner circle $\overline{\gamma_\varepsilon}$, which is a circle centered at a with radius ε, running clockwise (so that γ_ε runs counterclockwise). The "width" of the corridor is δ. See picture:



Hence $\Gamma_{\delta,\varepsilon}$ consists of four smooth curves.

Question 31.5.3. Draw a *simply connected* open set Ω which contains the entire $\Gamma_{\delta,\varepsilon}$ but does not contain the point a.

The function $\frac{f(z)}{z-a}$ manages to be holomorphic on all of Ω. Thus Cauchy's theorem applies and tells us that

$$0 = \oint_{\Gamma_{\delta,\varepsilon}} \frac{f(z)}{z-a}\,dz.$$

As we let $\delta \to 0$, the two walls of the keyhole will cancel each other (because f is continuous, and the walls run in opposite directions). So taking the limit as $\delta \to 0$, we are left with just γ and γ_ε, which (taking again orientation into account) gives

$$\oint_\gamma \frac{f(z)}{z-a}\,dz = -\oint_{\overline{\gamma_\varepsilon}} \frac{f(z)}{z-a}\,dz = \oint_{\gamma_\varepsilon} \frac{f(z)}{z-a}\,dz.$$

Thus **we've managed to replace γ with a much smaller circle γ_ε centered around** a, and the rest is algebra.

To compute the last quantity, write

$$\begin{aligned}\oint_{\gamma_\varepsilon} \frac{f(z)}{z-a}\,dz &= \oint_{\gamma_\varepsilon} \frac{f(z)-f(a)}{z-a}\,dz + f(a)\cdot\oint_{\gamma_\varepsilon} \frac{1}{z-a}\,dz \\ &= \oint_{\gamma_\varepsilon} \frac{f(z)-f(a)}{z-a}\,dz + 2\pi i f(a).\end{aligned}$$

where we've used Theorem 31.3.1. Thus, all we have to do is show that

$$\oint_{\gamma_\varepsilon} \frac{f(z)-f(a)}{z-a}\, dz = 0.$$

For this we can basically use the weakest bound possible, the so-called ML lemma which I'll cite without proof: it says "bound the function everywhere by its maximum".

Lemma 31.5.4 (ML estimation lemma)

Let f be a holomorphic function and α a path. Suppose $M = \max_{z \text{ on } \alpha} |f(z)|$, and let L be the length of α. Then

$$\left| \oint_\alpha f(z)\, dz \right| \le ML.$$

(This is straightforward to prove if you know the definition of length: $L = \int_a^b |\alpha'(t)|\, dt$, where $\alpha\colon [a,b] \to \mathbb{C}$.)

Anyways, as $\varepsilon \to 0$, the quantity $\frac{f(z)-f(a)}{z-a}$ approaches $f'(a)$, and so for small enough ε (i.e. z close to a) there's some upper bound M. Yet the length of γ_ε is the circumference $2\pi\varepsilon$. So the ML lemma says that

$$\left| \oint_{\gamma_\varepsilon} \frac{f(z)-f(a)}{z-a} \right| \le 2\pi\varepsilon \cdot M \to 0$$

as desired. □

§31.6 Holomorphic functions are analytic

Prototypical example for this section: Imagine a formal series $\sum_k c_k x^k$!

In the setup of the previous problem, we have a circle $\gamma\colon [0, 2\pi] \to \mathbb{C}$ and a holomorphic function $f\colon U \to \mathbb{C}$ which contains the disk D. We can write

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z-a}\, dz \\ &= \frac{1}{2\pi i} \oint_\gamma \frac{f(z)/z}{1-\frac{a}{z}}\, dz \\ &= \frac{1}{2\pi i} \oint_\gamma f(z)/z \cdot \sum_{k\ge 0} \left(\frac{a}{z}\right)^k\, dz \end{aligned}$$

You can prove (using the so-called Weierstrass M-test) that the summation order can be switched:

$$\begin{aligned} f(a) &= \frac{1}{2\pi i} \sum_{k\ge 0} \oint_\gamma \frac{f(z)}{z} \cdot \left(\frac{a}{z}\right)^k\, dz \\ &= \frac{1}{2\pi i} \sum_{k\ge 0} \oint_\gamma a^k \cdot \frac{f(z)}{z^{k+1}}\, dz \\ &= \sum_{k\ge 0} \left(\frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z^{k+1}}\, dz \right) a^k. \end{aligned}$$

Letting $c_k = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z^{k+1}} \, dz$, and noting this is independent of a, this is

$$f(a) = \sum_{k \ge 0} c_k a^k$$

and that's the miracle: holomorphic functions are given by a **Taylor series**! This is one of the biggest results in complex analysis. Moreover, if one is willing to believe that we can take the derivative k times, we obtain

$$c_k = \frac{f^{(k)}(0)}{k!}$$

and this gives us $f^{(k)}(0) = k! \cdot c_k$.

Naturally, we can do this with any circle (not just one centered at zero). So let's state the full result below, with arbitrary center p.

Theorem 31.6.1 (Cauchy's differentiation formula)
Let $f \colon U \to \mathbb{C}$ be a holomorphic function and let D be a disk centered at point p bounded by a circle γ. Suppose D is contained inside U. Then f is given everywhere in D by a Taylor series

$$f(z) = c_0 + c_1(z-p) + c_2(z-p)^2 + \cdots$$

where

$$c_k = \frac{f^k(p)}{k!} = \frac{1}{2\pi i} \oint_\gamma \frac{f(w-p)}{(w-p)^{k+1}} \, dw$$

In particular,

$$f^{(k)}(p) = k! c_k = \frac{k!}{2\pi i} \oint_\gamma \frac{f(w-p)}{(w-p)^{k+1}} \, dw.$$

Most importantly,

Over any disk, a holomorphic function is given exactly by a Taylor series.

This establishes a result we stated at the beginning of the chapter: that a function being complex differentiable once means it is not only infinitely differentiable, but in fact equal to its Taylor series.

Remark 31.6.2 — If you're willing to assume this, you can see why Cauchy-Goursat theorem should be true: assuming

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots$$

then, with γ the unit circle,

$$\begin{aligned} \oint_\gamma f(z) \, dz &= \oint_\gamma c_0 + c_1 z + c_2 z^2 + \ldots \, dz \\ &= \left(\oint_\gamma c_0 \, dz \right) + \left(\oint_\gamma c_1 z \, dz \right) + \left(\oint_\gamma c_2 z^2 \, dz \right) + \cdots \end{aligned}$$

We have already proven that each $\oint_\gamma z^m \, dz = 0$, so the sum ought to be 0 as well.

Of course the argument is not completely rigorous, it exchanges the integration and the infinite sum without justification.

Remark 31.6.3 — You can see where the term $\frac{f(w-p)}{(w-p)^{k+1}}$ comes from in Remark 32.3.5. It is very intuitive that even if you forget it, you can derive it yourself as well!

I should maybe emphasize a small subtlety of the result: the Taylor series centered at p is only valid in a disk centered at p which lies entirely in the domain U. If $U = \mathbb{C}$ this is no issue, since you can make the disk big enough to accommodate any point you want. It's more subtle in the case that U is, for example, a square; you can't cover the entire square with a disk centered at some point without going outside the square. However, since U is open we can at any rate at least find some open neighborhood for which the Taylor series is correct – in stark contrast to the real case. Indeed, as you'll see in the problems, the existence of a Taylor series is incredibly powerful.

§31.7 Optional: Proof that holomorphic functions are analytic

It is recommended to read the next chapter first to understand the origin of the term $\frac{f(w-p)}{(w-p)^{k+1}}$ in Cauchy's differentiation formula above.

Each step of the proof is quite intuitive, if not a bit long. The outline is:

- We pretend that the function f is analytic. (Yes, this is not circular reasoning!)
- We use Cauchy's differentiation formula to write down a power series:[3]

$$c_0 + c_1 z + c_2 z^2 + \cdots$$

- We prove that the power series coincide with f using Cauchy-Goursat theorem.
- Note that the statement "f is analytic" literally means "for every $k \geq 0$, then $f^{(k)}$ is differentiable". So, we write down a power series for $f^{(k)}$, and show that it is differentiable. (We already did this for the real case in Proposition 29.4.5.)

§31.7.i Proof of Cauchy-Goursat theorem

Suppose f is holomorphic i.e. differentiable. We wish to prove $\oint_\gamma f\, dz = 0$.

How may we attack this problem? Looking at the conclusion, we may want to stare at some function where $\oint_\gamma f\, dz \neq 0$.

We readily got an example from the previous chapter: $f(z) = \frac{1}{z}$.

Question 31.7.1. What part of the hypothesis does not hold?

In any case, you see the problem is it's because f has a singularity at 0 (even though we haven't formally defined what a singularity is yet). So, we try to prove the contrapositive:

Theorem 31.7.2

Suppose $\oint_\gamma f\, dz \neq 0$. Then something weird happens to f somewhere inside γ.

[3] Assume $0 \in U$.

(For arbitrary loops, it gets a bit more difficult, however. What does "inside γ" mean?)

Phrasing like this, it isn't that difficult. You may want to look at $f(z) = \frac{1}{z}$ a bit and try to figure out how the proof follows before continue reading.

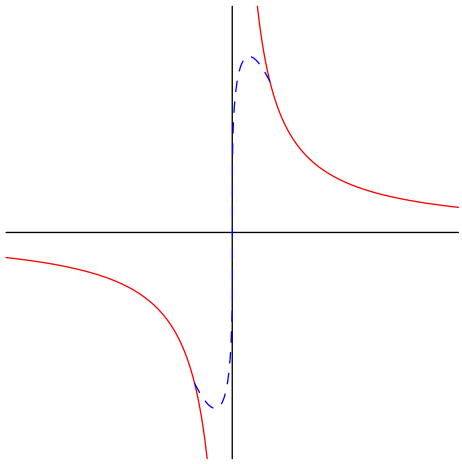
For simplicity, I will prove the statement for γ being a rectangle, leaving the case e.g. γ is a circle to the reader. The case of fully general γ will be handled later on.

As you may figured out, for $f(z) = \frac{1}{z-w}$, you can try to locate where the singularity w is by "binary search": compute $\oint_\gamma f\ dz$, if it is $2\pi i$, we know w is inside γ. We're going to do just that.

What should we search for? Let's see:

> **Exercise 31.7.3.** Suppose $\oint_\gamma f\ dz \neq 0$. Must there be a point where f blows up to infinity, like the point $z = 0$ in $\frac{1}{z}$?

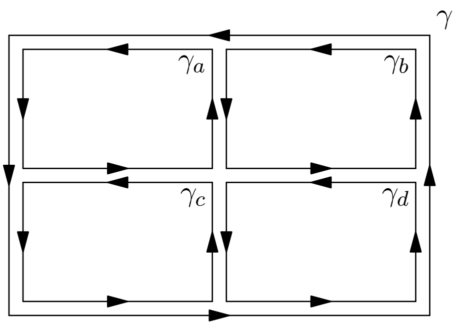
Answer: no, unfortunately. You can certainly take the function f above, and "smooth out" the singularity.



(Only real part depicted. You can imagine the imaginary part.)

The best we can hope for, then, is to find a point where f is not holomorphic (complex differentiable).

Construct 4 paths γ_a, γ_b, γ_c and γ_d as follows. The margin is only for illustration purpose, in reality the edges directly overlap on each other.



Notice that, because all the inner edges cancel out,

$$\oint_\gamma f\ dz = \oint_{\gamma_a} f\ dz + \oint_{\gamma_b} f\ dz + \oint_{\gamma_c} f\ dz + \oint_{\gamma_d} f\ dz.$$

Which means $\oint_{\gamma_i} f\ dz \neq 0$ for some $i \in \{a, b, c, d\}$. (Idea: we have more accurately located the singularity, now we know it is inside γ_i. Of course it's also possible that there are multiple singularities.)

We also have $|\oint_{\gamma_i} f\ dz| \ge \frac{1}{4} \cdot |\oint_\gamma f\ dz|$ for some i. The reason why we must carefully keep track of the magnitude (instead of just saying it's $\neq 0$) will become apparent later.

So, we keep doing that, and get a decreasing sequence of rectangles $\{\gamma_j\}$. Because the edge length gets halved each time, the rectangles converge to a single point p.

How would the rectangle perimeter decrease? Perhaps something like the following:

j	Perimeter of γ_j	$\lvert\oint_{\gamma_j} f\ dz\rvert$
0	1	1
1	$\frac{1}{2}$	$\ge \frac{1}{4}$
2	$\frac{1}{4}$	$\ge \frac{1}{16}$
3	$\frac{1}{8}$	$\ge \frac{1}{64}$

$|\oint_{\gamma_j} f\ dz|$ decreases quite quickly compared to the perimeter — as expected, we cannot hope for f to blow up at p, but this is sufficient to show f is not holomorphic.

For the sake of contradiction, assume otherwise. Then, by definition,

$$\lim_{h\to 0} \frac{f(p+h)-f(p)}{h} = f'(p)$$

where p is the point that the rectangles $\{\gamma_j\}$ converges to as defined above, and $f'(p) \in \mathbb{C}$ is the derivative. In other words, for $h \in \mathbb{C}$ close enough to 0,

$$f(p+h) = f(p) + f'(p)\cdot h + \varepsilon(h)\cdot h \text{ for } \varepsilon(h) \in o(1).$$

Why is this a problem? Notice that $f(p)$ and $f'(p)\cdot h$ are both polynomials, so

$$\oint_{\gamma_j} f(p) + f'(p)\cdot(z-p)\ dz = 0,$$

which means

$$\oint_{\gamma_j} f(z)\ dz = \oint_{\gamma_j} \varepsilon(h)\cdot(z-p)\ dz.$$

We know the left hand side decreases as 4^{-j}, but the integral on the right hand side is over a curve with length decreasing as 2^{-j}.

> **Exercise 31.7.4.** Finish the proof. (Use the ML estimation lemma.)

Finally, what to do with arbitrary curve (which may not even have an interior[4])?

We construct the antiderivative $F\colon \Omega \to \mathbb{C}$ by integrating f across the side of a rectangle, prove $F' = f$, and get a "fundamental theorem of calculus", that is

$$\oint_\alpha f(z)\ dz = F(\alpha(b)) - F(\alpha(a))$$

where $\alpha\colon [a,b] \to \mathbb{C}$ is some path. Considering $\alpha = \gamma$, because the starting and ending point for a loop γ is the same, of course the integral would be 0.

§31.7.ii The rest

Next step, we should show the power series coincide with f, that is

$$f(z) = \oint_\gamma \frac{f(t)}{t}\ dt + \oint_\gamma \frac{f(t)}{t^2}\ dt\cdot z + \oint_\gamma \frac{f(t)}{t^3}\ dt \cdot z^2 + \cdots$$

Here we assume γ is the unit circle, the power series is centered at 0, and t is inside the unit disk.

[4] A space-filling curve is an example.

Exercise 31.7.5. Prove it. (You only need to know that you can interchange the infinite sum and the integral in this situation,[a] how to sum a geometric series, and Cauchy's integral formula)

[a]Look at Example 38.1.4 for some horror stories where you cannot interchange a limit and an integral.

Remark 31.7.6 — *Wait, where was Cauchy-Goursat theorem used?* If you forgot, it is used in the proof of Cauchy's integral formula.

After we have proven that f is a power series, then using Proposition 29.4.5 (suitably adapted for the case of complex holomorphic functions), the result follows.

§31.8 A few harder problems to think about

These aren't olympiad problems, but I think they're especially nice! In the next complex analysis chapter we'll see some more nice applications.

The first few results are the most important.

Problem 31A* (Liouville's theorem)**.** Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose that $|f(z)| < 1000$ for all complex numbers z. Prove that f is a constant function.

Problem 31B* (Zeros are isolated)**.** An **isolated set** in an open set U in the complex plane is a set of points S such that around each point in S, one can draw an open neighborhood not intersecting any other point of S.

Show that the zero set of any nonzero holomorphic function $f\colon U \to \mathbb{C}$ is an isolated set, unless there exists a nonempty open subset of U on which f is identically zero.

Problem 31C* (Identity theorem)**.** Let $f, g\colon U \to \mathbb{C}$ be holomorphic, and assume that U is connected. Prove that if f and g agree on some open neighborhood, then $f = g$.

Problem 31D† (Maximums Occur On Boundaries)**.** Let $f\colon U \to \mathbb{C}$ be holomorphic, let $Y \subseteq U$ be compact, and let ∂Y be boundary[5] of Y. Show that

$$\max_{z \in Y} |f(z)| = \max_{z \in \partial Y} |f(z)| .$$

In other words, the maximum values of $|f|$ occur on the boundary. (Such maximums exist by compactness.)

Problem 31E (Harvard quals)**.** Let $f\colon \mathbb{C} \to \mathbb{C}$ be a nonconstant entire function. Prove that $f^{\text{img}}(\mathbb{C})$ is dense in $\mathbb{C}$. (In fact, a much stronger result is true: Little Picard's theorem says that the image of a nonconstant entire function omits at most one point.)

Problem 31F (Removable singularity theorem)**.** Let U be open, $p \in U$, and $f\colon U \setminus \{p\} \to \mathbb{C}$ be holomorphic. Suppose f is bounded. Show that $\lim_{z \to p} f(z)$ exists, and the extension $f\colon U \to \mathbb{C}$ is holomorphic at p.

[5]The boundary ∂Y is the set of points p such that no open neighborhood of p is contained in Y. It is also a compact set if Y is compact.

32 Meromorphic functions

§32.1 The second nicest functions on earth

If holomorphic functions are like polynomials, then *meromorphic* functions are like rational functions. Basically, a meromorphic function is a function of the form $\frac{A(z)}{B(z)}$ where $A, B \colon U \to \mathbb{C}$ are holomorphic and B is not zero. The most important example of a meromorphic function is $\frac{1}{z}$.

We are going to see that meromorphic functions behave like "almost-holomorphic" functions. Specifically, a meromorphic function A/B will be holomorphic at all points except the zeros of B (called *poles*). By the identity theorem, there cannot be too many zeros of B! So meromorphic functions can be thought of as "almost holomorphic" (like $\frac{1}{z}$, which is holomorphic everywhere but the origin). We saw that

$$\frac{1}{2\pi i} \oint_\gamma \frac{1}{z} \, dz = 1$$

for $\gamma(t) = e^{it}$ the unit circle. We will extend our results on contours to such situations.

It turns out that, instead of just getting $\oint_\gamma f(z) \, dz = 0$ like we did in the holomorphic case, the contour integrals will actually be used to *count the number of poles* inside the loop γ. It's ridiculous, I know.

§32.2 Meromorphic functions

Prototypical example for this section: $\frac{1}{z}$, with a pole of order 1 and residue 1 at $z = 0$.

Let U be an open subset of $\mathbb{C}$ again.

Definition 32.2.1. A function $f \colon U \to \mathbb{C}$ is **meromorphic** if there exists holomorphic functions $A, B \colon U \to \mathbb{C}$ with B not identically zero in any open neighborhood, and $f(z) = A(z)/B(z)$ whenever $B(z) \neq 0$.

Let's see how this function f behaves. If $z \in U$ has $B(z) \neq 0$, then in some small open neighborhood the function B isn't zero at all, and thus A/B is in fact *holomorphic*; thus f is holomorphic at z. (Concrete example: $\frac{1}{z}$ is holomorphic in any disk not containing 0.)

On the other hand, suppose $p \in U$ has $B(p) = 0$: without loss of generality, $p = 0$ to ease notation. By using the Taylor series at $p = 0$ we can put

$$B(z) = c_k z^k + c_{k+1} z^{k+1} + \cdots$$

with $c_k \neq 0$ (certainly some coefficient is nonzero since B is not identically zero!). Then we can write

$$\frac{1}{B(z)} = \frac{1}{z^k} \cdot \frac{1}{c_k + c_{k+1} z + \cdots}.$$

But the fraction on the right is a holomorphic function in this open neighborhood! So all that's happened is that we have an extra z^{-k} kicking around.

This gives us an equivalent way of viewing meromorphic functions:

Definition 32.2.2. Let $f: U \to \mathbb{C}$ as usual. A **meromorphic** function is a function which is holomorphic on U except at an isolated set S of points (meaning it is holomorphic as a function $U \setminus S \to \mathbb{C}$). For each $p \in S$, called a **pole** of f, the function f is further required to admit a **Laurent series**, meaning that

$$f(z) = \frac{c_{-m}}{(z-p)^m} + \frac{c_{-m+1}}{(z-p)^{m-1}} + \dots + \frac{c_{-1}}{z-p} + c_0 + c_1(z-p) + \dots$$

for all z in some open neighborhood of p, other than $z = p$. Here m is a positive integer, and $c_{-m} \neq 0$.

Note that the trailing end *must* terminate. By "isolated set", I mean that we can draw open neighborhoods around each pole in S, in such a way that no two open neighborhoods intersect.

Example 32.2.3 (Example of a meromorphic function)

Consider the function

$$\frac{z+1}{\sin z}.$$

It is meromorphic, because it is holomorphic everywhere except at the zeros of $\sin z$. At each of these points we can put a Laurent series: for example at $z = 0$ we have

$$\begin{aligned} \frac{z+1}{\sin z} &= (z+1) \cdot \frac{1}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots} \\ &= \frac{1}{z} \cdot \frac{z+1}{1 - \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} - \dots\right)} \\ &= \frac{1}{z} \cdot (z+1) \sum_{k \ge 0} \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^6}{7!} - \dots\right)^k. \end{aligned}$$

If we expand out the horrible sum (which I won't do), then you get $\frac{1}{z}$ times a perfectly fine Taylor series, i.e. a Laurent series.

Abuse of Notation 32.2.4. We'll often say something like "consider the function $f: \mathbb{C} \to \mathbb{C}$ by $z \mapsto \frac{1}{z}$". Of course this isn't completely correct, because f doesn't have a value at $z = 0$. If I was going to be completely rigorous I would just set $f(0) = 2015$ or something and move on with life, but for all intents let's just think of it as "undefined at $z = 0$".

Why don't I just write $g: \mathbb{C} \setminus \{0\} \to \mathbb{C}$? The reason I have to do this is that it's still important for f to remember it's "trying" to be holomorphic on $\mathbb{C}$, even if isn't assigned a value at $z = 0$. As a function $\mathbb{C} \setminus \{0\} \to \mathbb{C}$ the function $\frac{1}{z}$ is actually holomorphic.

Remark 32.2.5 — I have shown that any function $A(z)/B(z)$ has this characterization with poles, but an important result is that the converse is true too: if $f: U \setminus S \to \mathbb{C}$ is holomorphic for some isolated set S, and moreover f admits a Laurent series at each point in S, then f can be written as a rational quotient of holomorphic functions. I won't prove this here, but it is good to be aware of.

Definition 32.2.6. Let p be a pole of a meromorphic function f, with Laurent series

$$f(z) = \frac{c_{-m}}{(z-p)^m} + \frac{c_{-m+1}}{(z-p)^{m-1}} + \dots + \frac{c_{-1}}{z-p} + c_0 + c_1(z-p) + \dots.$$

The integer m is called the **order** of the pole. A pole of order 1 is called a **simple pole**. We also give the coefficient c_{-1} a name, the **residue** of f at p, which we write $\operatorname{Res}(f; p)$.

The order of a pole tells you how "bad" the pole is. The order of a pole is the "opposite" concept of the **multiplicity** of a **zero**. If f has a pole at zero, then its Laurent series near $z = 0$ might look something like

$$f(z) = \frac{1}{z^5} + \frac{8}{z^3} - \frac{2}{z^2} + \frac{4}{z} + 9 - 3z + 8z^2 + \cdots$$

and so f has a pole of order five. By analogy, if g has a zero at $z = 0$, it might look something like

$$g(z) = 3z^3 + 2z^4 + 9z^5 + \cdots$$

and so g has a zero of multiplicity three. These orders are additive: $f(z)g(z)$ still has a pole of order $5 - 3 = 2$, but $f(z)g(z)^2$ is completely patched now, and in fact has a **simple zero** now (that is, a zero of degree 1).

> **Exercise 32.2.7.** Convince yourself that orders are additive as described above. (This is obvious once you understand that you are multiplying Taylor/Laurent series.)

Metaphorically, poles can be thought of as "negative zeros".

We can now give many more examples.

Example 32.2.8 (Examples of meromorphic functions)

(a) Any holomorphic function is a meromorphic function which happens to have no poles. Stupid, yes.

(b) The function $\mathbb{C} \to \mathbb{C}$ by $z \mapsto 100z^{-1}$ for $z \neq 0$ but undefined at zero is a meromorphic function. Its only pole is at zero, which has order 1 and residue 100.

(c) The function $\mathbb{C} \to \mathbb{C}$ by $z \mapsto z^{-3} + z^2 + z^9$ is also a meromorphic function. Its only pole is at zero, and it has order 3, and residue 0.

(d) The function $\mathbb{C} \to \mathbb{C}$ by $z \mapsto \frac{e^z}{z^2}$ is meromorphic, with the Laurent series at $z = 0$ given by

$$\frac{e^z}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2} + \frac{z}{6} + \frac{z^2}{24} + \frac{z^3}{120} + \cdots.$$

Hence the pole $z = 0$ has order 2 and residue 1.

Example 32.2.9 (A rational meromorphic function)

Consider the function $\mathbb{C} \to \mathbb{C}$ given by

$$\begin{aligned} z \mapsto \frac{z^4+1}{z^2-1} &= z^2 + 1 + \frac{2}{(z-1)(z+1)} \\ &= z^2 + 1 + \frac{1}{z-1} \cdot \frac{1}{1 + \frac{z-1}{2}} \\ &= \frac{1}{z-1} + \frac{3}{2} + \frac{9}{4}(z-1) + \frac{7}{8}(z-1)^2 - \ldots \end{aligned}$$

It has a pole of order 1 and residue 1 at $z = 1$. (It also has a pole of order 1 at

$z = -1$; you are invited to compute the residue.)

Example 32.2.10 (Function with infinitely many poles)
The function $\mathbb{C} \to \mathbb{C}$ by
$$z \mapsto \frac{1}{\sin(z)}$$
has infinitely many poles: the numbers $z = \pi k$, where k is an integer. Let's compute the Laurent series at just $z = 0$:
$$\begin{aligned} \frac{1}{\sin(2\pi z)} &= \frac{1}{\frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots} \\ &= \frac{1}{z} \cdot \frac{1}{1 - \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots\right)} \\ &= \frac{1}{z} \sum_{k \ge 0} \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots\right)^k. \end{aligned}$$
which is a Laurent series, though I have no clue what the coefficients are. You can at least see the residue; the constant term of that huge sum is 1, so the residue is 1. Also, the pole has order 1.

Example 32.2.11 (A function that is not meromorphic)
Consider the function
$$z \mapsto \frac{1}{\sin(1/z)}.$$
It is a holomorphic function on
$$U = \mathbb{C} \setminus \{0\} \setminus S$$
where we define $S = \{\frac{1}{\pi k} \mid k \in \mathbb{Z} \setminus \{0\}\}$. Similar to $z \mapsto \frac{1}{\sin(z)}$, each point in the set S has a pole of order 1.

However, at $z = 0$, the function admits no Laurent series — if it were, there would be a neighborhood around $z = 0$ where the function is defined, but there is no such set.

However, f is meromorphic on $\mathbb{C} \setminus \{0\}$ — the set S is isolated, but $S \cup \{0\}$ is not isolated.

The Laurent series, if it exists, is unique (as you might have guessed), and by our result on holomorphic functions it is actually valid for *any* disk centered at p (minus the point p). The part $\frac{c_{-1}}{z-p} + \cdots + \frac{c_{-m}}{(z-p)^m}$ is called the **principal part**, and the rest of the series $c_0 + c_1(z-p) + \cdots$ is called the **analytic part**.

§32.3 Winding numbers and the residue theorem

Recall that for a counterclockwise circle γ and a point p inside it, we had
$$\oint_\gamma (z-p)^m \, dz = \begin{cases} 0 & m \neq -1 \\ 2\pi i & m = -1 \end{cases}$$

where m is an integer. One can extend this result to in fact show that $\oint_\gamma (z-p)^m \, dz = 0$ for *any* loop γ, where $m \neq -1$. So we associate a special name for the nonzero value at $m = -1$.

Definition 32.3.1. For a point $p \in \mathbb{C}$ and a loop γ not passing through it, we define the **winding number**, denoted $\mathbf{I}(\gamma, p)$, by

$$\mathbf{I}(\gamma, p) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{z-p} \, dz$$

For example, by our previous results we see that if γ is a circle, we have

$$\mathbf{I}(\text{circle}, p) = \begin{cases} 1 & p \text{ inside the circle} \\ 0 & p \text{ outside the circle.} \end{cases}$$

If you've read the chapter on fundamental groups, then this is just the fundamental group associated to $\mathbb{C} \setminus \{p\}$. In particular, the winding number is always an integer. (Essentially, it uses the complex logarithm to track how the argument of the function changes. The details are more complicated, so we omit them here). In the simplest case the winding numbers are either 0 or 1.

Definition 32.3.2. We say a loop γ is **regular** if $\mathbf{I}(\gamma, p) = 1$ for all points p in the interior of γ (for example, if γ is a counterclockwise circle).

With all these ingredients we get a stunning generalization of the Cauchy-Goursat theorem:

Theorem 32.3.3 (Cauchy's residue theorem)

Let $f \colon \Omega \to \mathbb{C}$ be meromorphic, where Ω is simply connected. Then for any loop γ not passing through any of its poles, we have

$$\frac{1}{2\pi i} \oint_\gamma f(z) \, dz = \sum_{\text{pole } p} \mathbf{I}(\gamma, p) \operatorname{Res}(f; p).$$

In particular, if γ is regular then the contour integral is the sum of all the residues, in the form

$$\frac{1}{2\pi i} \oint_\gamma f(z) \, dz = \sum_{\substack{\text{pole } p \\ \text{inside } \gamma}} \operatorname{Res}(f; p).$$

> **Question 32.3.4.** Verify that this result coincides with what you expect when you integrate $\oint_\gamma cz^{-1} \, dz$ for γ a counter-clockwise circle.

The proof from here is not really too impressive – the "work" was already done in our statements about the winding number.

Proof. Let the poles with nonzero winding number be $p_1, \dots, p_k$ (the others do not affect the sum).[1] Then we can write f in the form

$$f(z) = g(z) + \sum_{i=1}^{k} P_i \left(\frac{1}{z - p_i} \right)$$

[1]To show that there must be finitely many such poles: recall that all our contours $\gamma \colon [a, b] \to \mathbb{C}$ are in fact bounded, so there is some big closed disk D which contains all of γ. The poles outside D thus have winding number zero. Now we cannot have infinitely many poles inside the disk D, for D is compact and the set of poles is a closed and isolated set!

where $P_i\left(\frac{1}{z-p_i}\right)$ is the principal part of the pole p_i. (For example, if $f(z) = \frac{z^3-z+1}{z(z+1)}$ we would write $f(z) = (z-1) + \frac{1}{z} - \frac{1}{1+z}$.)

The point of doing so is that the function g is holomorphic (we've removed all the "bad" parts), so

$$\oint_\gamma g(z)\, dz = 0$$

by Cauchy-Goursat.

On the other hand, if $P_i(x) = c_1 x + c_2 x^2 + \dots + c_d x^d$ then

$$\begin{aligned}\oint_\gamma P_i\left(\frac{1}{z-p_i}\right)\, dz &= \oint_\gamma c_1 \cdot \left(\frac{1}{z-p_i}\right)\, dz + \oint_\gamma c_2 \cdot \left(\frac{1}{z-p_i}\right)^2\, dz + \dots \\ &= c_1 \cdot \mathbf{I}(\gamma, p_i) + 0 + 0 + \dots \\ &= \mathbf{I}(\gamma, p_i) \operatorname{Res}(f; p_i).\end{aligned}$$

which gives the conclusion. $\square$

Remark 32.3.5 (Intuition behind Cauchy's integral formula) — In the setting of Theorem 31.5.1, note that if f is meromorphic in the disk D, we can compute the Laurent series of f at the point a:

$$f(z) = \frac{c_{-m}}{(z-a)^m} + \frac{c_{-m+1}}{(z-a)^{m-1}} + \dots + \frac{c_{-1}}{z-a} + c_0 + c_1(z-a) + \cdots$$

By the residue theorem, integrating $f(z)$ around the boundary of D results in the c_{-1} coefficient in the Laurent series:

$$\frac{1}{2\pi i} \oint_\gamma f(z)\, dz = \operatorname{Res}(f; a) = c_{-1}.$$

Of course, this is useless — f is holomorphic at a, so $c_{-1} = 0$. We want to compute $c_0 = f(a)$ instead.

Nevertheless, the trick is that *we can manipulate the function* f in order to move the coefficient we want to compute to the coefficient corresponding to $(z-a)^{-1}$. How are we going to do that? By dividing by $z-a$, of course!

So, $\frac{f(z)}{z-a}$ is meromorphic in the disk D, with Laurent series expansion around a being

$$\frac{f(z)}{z-a} = \frac{c_{-m}}{(z-a)^{m+1}} + \frac{c_{-m+1}}{(z-a)^m} + \dots + \frac{c_{-1}}{(z-a)^2} + \frac{c_0}{z-a} + c_1 + c_2(z-a) + \cdots$$

Because $\frac{f(z)}{z-a}$ has no other poles in D except at a, the residue theorem immediately tells us the integral $\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-a}\, dz$ equals $\operatorname{Res}(\frac{f(z)}{z-a}; a)$, which equals c_0 looking at the Laurent series above.

§32.4 Argument principle

One tricky application is as follows. Given a polynomial $P(x) = (x-a_1)^{e_1}(x-a_2)^{e_2} \dots (x-a_n)^{e_n}$, you might know that we have

$$\frac{P'(x)}{P(x)} = \frac{e_1}{x-a_1} + \frac{e_2}{x-a_2} + \dots + \frac{e_n}{x-a_n}.$$

The quantity P'/P is called the **logarithmic derivative**, as it is the derivative of $\log P$. This trick allows us to convert zeros of P into poles of P'/P with order 1; moreover the residues of these poles are the multiplicities of the roots.

In an analogous fashion, we can obtain a similar result for any meromorphic function f.

Proposition 32.4.1 (The logarithmic derivative)
Let $f : U \to \mathbb{C}$ be a meromorphic function. Then the logarithmic derivative f'/f is meromorphic as a function from U to $\mathbb{C}$; its only poles are:

(i) A pole at each zero of f whose residue is the multiplicity, and

(ii) A pole at each pole of f whose residue is the negative of the pole's order.

Again, you can almost think of a pole as a zero of negative multiplicity. This spirit is exemplified below.

Proof. Dead easy with Laurent series. Let a be a zero/pole of f, and WLOG set $a = 0$ for convenience. We take the Laurent series at zero to get

$$f(z) = c_k z^k + c_{k+1} z^{k+1} + \dots$$

where $k < 0$ if 0 is a pole and $k > 0$ if 0 is a zero. Taking the derivative gives

$$f'(z) = kc_k z^{k-1} + (k+1)c_{k+1} z^k + \dots.$$

Now look at f'/f; with some computation, it equals

$$\frac{f'(z)}{f(z)} = \frac{1}{z} \frac{kc_k + (k+1)c_{k+1} z + \dots}{c_k + c_{k+1} z + \dots}.$$

So we get a simple pole at $z = 0$, with residue k. □

Using this trick you can determine the number of zeros and poles inside a regular closed curve, using the so-called Argument Principle.[2]

Theorem 32.4.2 (Argument principle)
Let γ be a regular curve. Suppose $f : U \to \mathbb{C}$ is meromorphic inside and on γ, and none of its zeros or poles lie on γ. Then

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'}{f} \, dz = \frac{1}{2\pi i} \oint_{f \circ \gamma} \frac{1}{z} \, dz = Z - P$$

where Z is the number of zeros inside γ (counted with multiplicity) and P is the number of poles inside γ (again with multiplicity).

[2] So-called because the *argument* of a complex number z is the angle formed by the real axis and the vector representing z, not because you need to use any argument. If $z \in \mathbb{C}$ is interpreted as a point in $\mathbb{R}^2$, the argument of z is the same as $\theta(z)$ defined in Example 44.7.4.

Proof. Immediate by applying Cauchy's residue theorem alongside the preceding proposition. In fact you can generalize to any curve γ via the winding number: the integral is

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'}{f}\,dz = \sum_{\text{zero } z} \mathbf{I}(\gamma, z) - \sum_{\text{pole } p} \mathbf{I}(\gamma, p)$$

where the sums are with multiplicity. $\square$

Thus the Argument Principle allows one to count zeros and poles inside any region of choice.

Computers can use this to get information on functions whose values can be computed but whose behavior as a whole is hard to understand. Suppose you have a holomorphic function f, and you want to understand where its zeros are. Then just start picking various circles γ. Even with machine rounding error, the integral will be close enough to the true integer value that we can decide how many zeros are in any given circle. Numerical evidence for the Riemann Hypothesis (concerning the zeros of the Riemann zeta function) can be obtained in this way.

§32.5 Digression: the Argument Principle viewed geometrically

There is another, more geometric, way to understand the Argument Principle.

Assume a function f is holomorphic on a connected open set U containing 0, and possibly has a zero or a pole at 0. Let $\gamma\colon [0, 2\pi] \to U$ be some curve contained in U, such that 0 is not in the image of the curve.

Let $a = \gamma(0)$ be the starting point of γ, and $b = \gamma(2\pi)$ be the ending point of γ.

We all know that $z \mapsto \log z$ is not an actual function — even ignoring the singularity at 0, it has a branch cut (we will formally handle this in Chapter 33).

Nevertheless, if we close our eyes and shuffling some symbols around, we get:

$$\begin{aligned}
\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz &= \frac{1}{2\pi i}\oint_\gamma \frac{d}{dz}\log f(z)\,dz \\
&= \frac{1}{2\pi i}\oint_\gamma d(\log f(z)) \\
&= \frac{1}{2\pi i}\cdot(\log f(b) - \log f(a)).
\end{aligned}$$

Miraculously, everything seems to cancel out so nicely! This is not a coincidence.

Now, if γ is a circle, then $a = b$, so the formula above seemingly states that the integral will be 0? Fortunately for us, no — log is in fact not a function.

So, does the formula above means anything? It does! While we won't prove this rigorously, the point is that:

If we let a point p smoothly moves from a to b, and let $\log f(p)$ follows the value, then $\log f(b) - \log f(a)$ represents the change in value of $\log f(p)$.

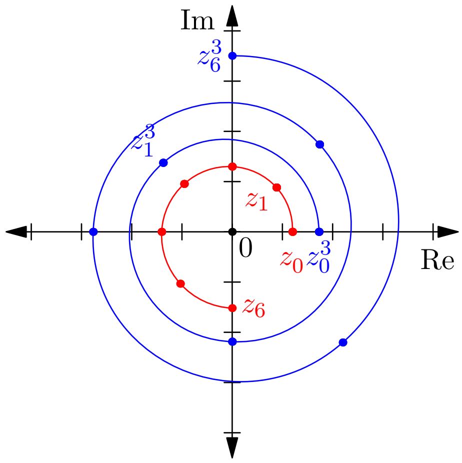
In the notation of Section 66.2, we have the mouse moves along γ from a to b, the first robot moves along $f \circ \gamma$ from $f(a)$ to $f(b)$, and the second robot moves from $\log f(a)$ to $\log f(b)$.

If we forget about the mouse for a moment, note that:

The quantity $\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz$ is equal to the number of times the *first* robot winds around the origin.

That is, $\mathbf{I}(f\circ\gamma, 0)$. (This is essentially obvious to see, because of all the work we have done to prove $\oint d\log z = \oint \frac{1}{z}\,dz$ equals the winding number.)

Finally, if we look at some simple examples like z^3:



We can immediately see the relation between the winding number and the multiplicity of a zero:

If z moves around the origin in a circle once, then z^n moves around the origin in a circle n times.

z^{-n} is not much different — it moves around the origin in a circle n times, just in the opposite direction.

Piecing all these pieces together, we get the Argument Principle — the logarithmic derivative can be used to count the multiplicity of the roots and the order of the poles.

§32.6 Philosophy: why are holomorphic functions so nice?

All the fun we've had with holomorphic and meromorphic functions comes down to the fact that complex differentiability is such a strong requirement. It's a small miracle that $\mathbb{C}$, which *a priori* looks only like $\mathbb{R}^2$, is in fact a field. Moreover, $\mathbb{R}^2$ has the nice property that one can draw nontrivial loops (it's also true for real functions that $\int_a^a f\,dx = 0$, but this is not so interesting!), and this makes the theory much more interesting.

As another piece of intuition from Siu[3]: If you try to get (left) differentiable functions over *quaternions*, you find yourself with just linear functions.

§32.7 A few harder problems to think about

Problem 32A (Fundamental theorem of algebra)**.** Prove that if f is a nonzero polynomial of degree n then it has n roots.

Problem 32B† (Rouché's theorem)**.** Let $f, g\colon U \to \mathbb{C}$ be holomorphic functions, where U contains the unit disk. Suppose that $|f(z)| > |g(z)|$ for all z on the unit circle. Prove

[3] Harvard professor.

that f and $f+g$ have the same number of zeros which lie strictly inside the unit circle (counting multiplicities).

Problem 32C (Wedge contour)**.** For each odd integer $n \geq 3$, evaluate the improper integral

$$\int_0^\infty \frac{1}{1+x^n}\,dx.$$

Problem 32D (Another contour)**.** Prove that the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx$$

converges and determine its value.

Problem 32E*. Let $f \colon U \to \mathbb{C}$ be a nonconstant holomorphic function.

(a) (Open mapping theorem) Prove that $f^{\text{img}}(U)$ is open in $\mathbb{C}$.[4]

(b) (Maximum modulus principle) Show that $|f|$ cannot have a maximum over U. That is, show that for any $z \in U$, there is some $z' \in U$ such that $|f(z)| < |f(z')|$.

[4] Thus the image of *any* open set $V \subseteq U$ is open in $\mathbb{C}$ (by repeating the proof for the restriction of f to V).

33 Holomorphic square roots and logarithms

In this chapter we'll make sense of a holomorphic square root and logarithm. The main results are Theorem 33.3.2, Theorem 33.4.2, Corollary 33.5.1, and Theorem 33.5.2. If you like, you can read just these four results, and skip the discussion of how they came to be.

Let $f: U \to \mathbb{C}$ be a holomorphic function. A **holomorphic nth root** of f is a function $g: U \to \mathbb{C}$ such that $f(z) = g(z)^n$ for all $z \in U$. A **logarithm** of f is a function $g: U \to \mathbb{C}$ such that $f(z) = e^{g(z)}$ for all $z \in U$. The main question we'll try to figure out is: when do these exist? In particular, what if $f = \mathrm{id}$?

§33.1 Motivation: square root of a complex number

To start us off, can we define $\sqrt{z}$ for any complex number z?

The first obvious problem that comes up is that for any z, there are *two* numbers w such that $w^2 = z$. How can we pick one to use? For our ordinary square root function, we had a notion of "positive", and so we simply took the positive root.

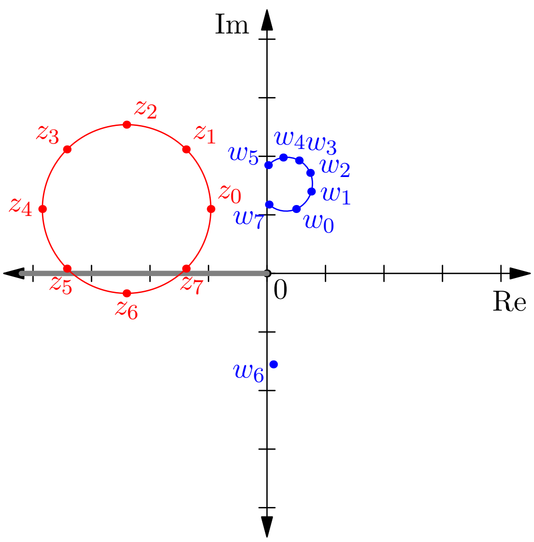
Let's expand on this: given $z = r(\cos\theta + i\sin\theta)$ (here $r \ge 0$) we should take the root to be

$$w = \sqrt{r}\,(\cos\alpha + i\sin\alpha)\,.$$

such that $2\alpha \equiv \theta \pmod{2\pi}$; there are two choices for $\alpha \pmod{2\pi}$, differing by π.

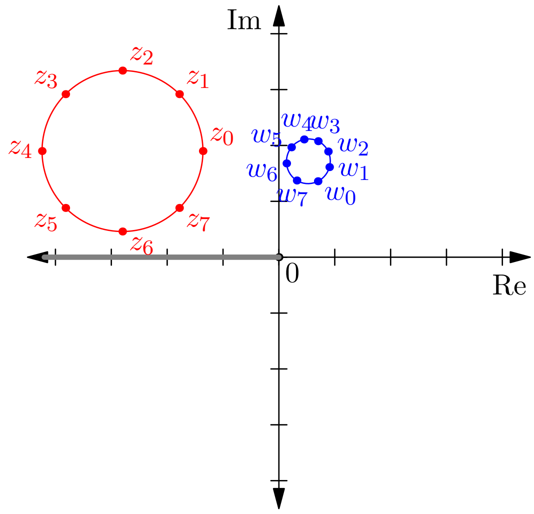
For complex numbers, we don't have an obvious way to pick α. Nonetheless, perhaps we can also get away with an arbitrary distinction: let's see what happens if we just choose the α with $-\frac{1}{2}\pi < \alpha \le \frac{1}{2}\pi$.

Pictured below are some points (in red) and their images (in blue) under this "upper-half" square root. The condition on α means we are forcing the blue points to lie on the right-half plane.

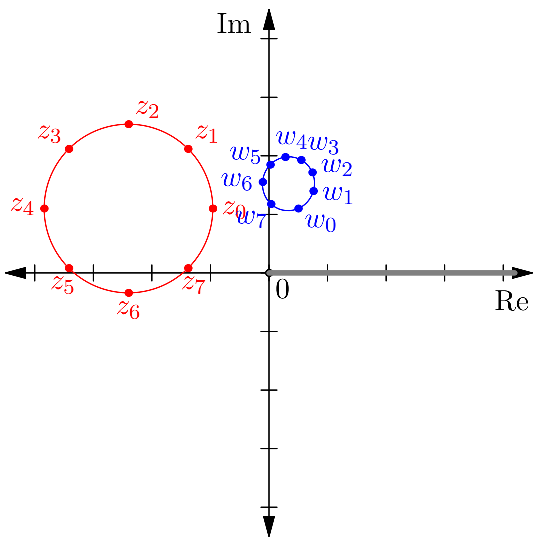


Here, $w_i^2 = z_i$ for each i, and we are constraining the w_i to lie in the right half of the complex plane. We see there is an obvious issue: there is a big discontinuity near the points w_5 and w_7! The nearby point w_6 has been mapped very far away. This discontinuity occurs since the points on the negative real axis are at the "boundary". For example, given -4, we send it to $-2i$, but we have hit the boundary: in our interval $-\frac{1}{2}\pi \le \alpha < \frac{1}{2}\pi$, we are at the very left edge.

The negative real axis that we must not touch is what we will later call a *branch cut*, but for now I call it a **ray of death**. It is a warning to the red points: if you cross this line, you will die! However, if we move the red circle just a little upwards (so that it misses the negative real axis) this issue is avoided entirely, and we get what seems to be a "nice" square root.



In fact, the ray of death is fairly arbitrary: it is the set of "boundary issues" that arose when we picked $-\frac{1}{2}\pi < \alpha \le \frac{1}{2}\pi$. Suppose we instead insisted on the interval $0 \le \alpha < \pi$; then the ray of death would be the *positive* real axis instead. The earlier circle we had now works just fine.



What we see is that picking a particular α-interval leads to a different set of edge cases, and hence a different ray of death. The only thing these rays have in common is their starting point of zero. In other words, given a red circle and a restriction of α, I can make a nice "square rooted" blue circle as long as the ray of death misses it.

So, what exactly is going on?

§33.2 Square roots of holomorphic functions

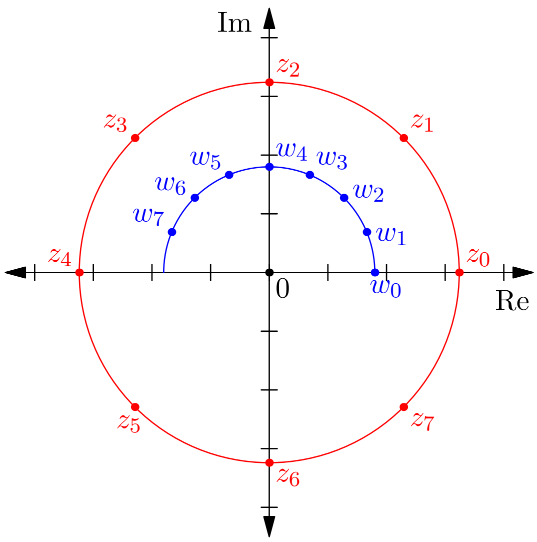
To get a picture of what's happening, we would like to consider a more general problem: let $f : U \to \mathbb{C}$ be holomorphic. Then we want to decide whether there is a holomorphic $g : U \to \mathbb{C}$ such that

$$f(z) = g(z)^2.$$

Our previous discussion with $f = \mathrm{id}$ tells us we cannot hope to achieve this for $U = \mathbb{C}$; there is a "half-ray" which causes problems. However, there are certainly functions $f : \mathbb{C} \to \mathbb{C}$ such that a g exists. As a simplest example, $f(z) = z^2$ should definitely have a square root!

Now let's see if we can fudge together a square root. Earlier, what we did was try to specify a rule to force one of the two choices at each point. This is unnecessarily strict. Perhaps we can do something like: start at a point in $z_0 \in U$, pick a square root w_0 of $f(z_0)$, and then try to "fudge" from there the square roots of the other points. What do I mean by fudge? Well, suppose z_1 is a point very close to z_0, and we want to pick a square root w_1 of $f(z_1)$. While there are two choices, we also would expect w_0 to be close to w_1. Unless we are highly unlucky, this should tell us which choice of w_1 to pick. (Stupid concrete example: if I have taken the square root $-4.12i$ of -17 and then ask you to continue this square root to -16, which sign should you pick for $\pm 4i$?)

There are two possible ways we could get unlucky in the scheme above: first, if $w_0 = 0$, then we're sunk. But even if we avoid that, we have to worry that if we run a full loop in the complex plane, we might end up in a different place from where we started. For concreteness, consider the following situation, again with $f = \mathrm{id}$:



We started at the point z_0, with one of its square roots as w_0. We then wound a full red circle around the origin, only to find that at the end of it, the blue arc is at a different place where it started!

The interval construction from earlier doesn't work either: no matter how we pick the interval for α, any ray of death must hit our red circle. The problem somehow lies with the fact that we have enclosed the very special point 0.

Nevertheless, we know that if we take $f(z) = z^2$, then we don't run into any problems with our "make it up as you go" procedure. So, what exactly is going on?

§33.3 Covering projections

By now, if you have read the part on algebraic topology, this should all seem quite familiar. The "fudging" procedure exactly describes the idea of a lifting.

More precisely, recall that there is a covering projection

$$(-)^2 \colon \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}.$$

Let $V = \{z \in U \mid f(z) \neq 0\}$. For $z \in U \setminus V$, we already have the square root $g(z) = \sqrt{f(z)} = \sqrt{0} = 0$. So the burden is completing $g \colon V \to \mathbb{C}$.

Then essentially, what we are trying to do is construct a lifting g in the diagram

$$\begin{tikzcd} & E = \mathbb{C} \setminus \{0\} \arrow[d, "p = \bullet^2"] \\ V \arrow[ur, "g"] \arrow[r, "f"'] & B = \mathbb{C} \setminus \{0\}. \end{tikzcd}$$

Our map p can be described as "winding around twice". Our Theorem 66.2.5 now tells us that this lifting exists if and only if

$$f_*^{\text{img}}(\pi_1(V)) \subseteq p_*^{\text{img}}(\pi_1(E))$$

is a subset of the image of $\pi_1(E)$ by p. Since B and E are both punctured planes, we can identify them with S^1.

> **Question 33.3.1.** Show that the image under p is exactly $2\mathbb{Z}$ once we identify $\pi_1(B) = \mathbb{Z}$.

That means that for any loop γ in V, we need $f \circ \gamma$ to have an *even* winding number around $0 \in B$. This amounts to

$$\frac{1}{2\pi} \oint_\gamma \frac{f'}{f} \, dz \in 2\mathbb{Z}$$

since f has no poles.

Replacing 2 with n and carrying over the discussion gives the first main result.

Theorem 33.3.2 (Existence of holomorphic nth roots)
Let $f \colon U \to \mathbb{C}$ be holomorphic. Then f has a holomorphic nth root if and only if

$$\frac{1}{2\pi i} \oint_\gamma \frac{f'}{f} \, dz \in n\mathbb{Z}$$

for every contour γ in U.

§33.4 Complex logarithms

The multivalued nature of the complex logarithm comes from the fact that

$$\exp(z + 2\pi i) = \exp(z).$$

So if $e^w = z$, then any complex number $w + 2\pi i k$ is also a solution.

We can handle this in the same way as before: it amounts to a lifting of the following diagram.

$$\begin{tikzcd} & E = \mathbb{C} \arrow[d, "p=\exp"] \\ U \arrow[ur, "g"] \arrow[r, "f"'] & B = \mathbb{C} \setminus \{0\} \end{tikzcd}$$

There is no longer a need to work with a separate V since:

Question 33.4.1. Show that if f has any zeros then g can't possibly exist.

In fact, the map $\exp\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ is a universal cover, since $\mathbb{C}$ is simply connected. Thus, $p^{\text{img}}(\pi_1(\mathbb{C}))$ is *trivial.* So in addition to being zero-free, f cannot have any winding number around $0 \in B$ at all. In other words:

Theorem 33.4.2 (Existence of logarithms)
Let $f\colon U \to \mathbb{C}$ be holomorphic. Then f has a logarithm if and only if

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'}{f}\,dz = 0$$

for every contour γ in U.

§33.5 Some special cases

The most common special case is

Corollary 33.5.1 (Nonvanishing functions from simply connected domains)
Let $f\colon \Omega \to \mathbb{C}$ be continuous, where Ω is simply connected. If $f(z) \neq 0$ for every $z \in \Omega$, then f has both a logarithm and holomorphic nth root.

Finally, let's return to the question of $f = \text{id}$ from the very beginning. What's the best domain U such that

$$\sqrt{-}\colon U \to \mathbb{C}$$

is well-defined? Clearly $U = \mathbb{C}$ cannot be made to work, but we can do almost as well. For note that the only zero of $f = \text{id}$ is at the origin. Thus if we want to make a logarithm exist, all we have to do is make an incision in the complex plane that renders it impossible to make a loop around the origin. The usual choice is to delete negative half of the real axis, our very first ray of death; we call this a **branch cut**, with **branch point** at $0 \in \mathbb{C}$ (the point which we cannot circle around). This gives

Theorem 33.5.2 (Branch cut functions)
There exist holomorphic functions

$$\begin{aligned} \log &: \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C} \\ \sqrt[n]{-} &: \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C} \end{aligned}$$

satisfying the obvious properties.

There are many possible choices of such functions (n choices for the nth root and infinitely many for log); a choice of such a function is called a **branch**. So this is what is meant by a "branch" of a logarithm.

The **principal branch** is the "canonical" branch, analogous to the way we arbitrarily pick the positive branch to define $\sqrt{-}\colon \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$. For log, we take the w such that $e^w = z$ and the imaginary part of w lies in $(-\pi, \pi]$ (since we can shift by integer multiples of $2\pi i$). Often, authors will write $\operatorname{Log} z$ to emphasize this choice.

§33.6 A few harder problems to think about

Problem 33A. Show that a holomorphic function $f\colon U \to \mathbb{C}$ has a holomorphic logarithm if and only if it has a holomorphic nth root for every integer n.

Problem 33B. Show that the function $f\colon U \to \mathbb{C}$ by $z \mapsto z(z-1)$ has a holomorphic square root, where U is the entire complex plane minus the closed interval $[0, 1]$.

34 Bonus: Topological Abel-Ruffini Theorem

We've already shown the Fundamental Theorem of Algebra. Now, with our earlier intuition on holomorphic nth roots, we can now show that there is no general formula for the roots of a quintic polynomial.

§34.1 The Game Plan

Firstly, what do we even mean by "formula" here?

Definition 34.1.1. A **quintic formula** would be a formula taking in the coefficients $(a_0, \ldots, a_5)$ of a degree 5 polynomial P, using the operations $+$, $-$, $\times$, $\div$, $\sqrt[n]{}$ finitely many times, that maps to the five roots $(z_1, \cdots, z_5)$ of P.

Now, any proposed quintic formula F receives the same coefficients when the roots are the same, and thus gives the same output. This is fine at first glance, but swapping two roots continuously might pose more issues. F must create and preserve some order of the roots under these permutations.

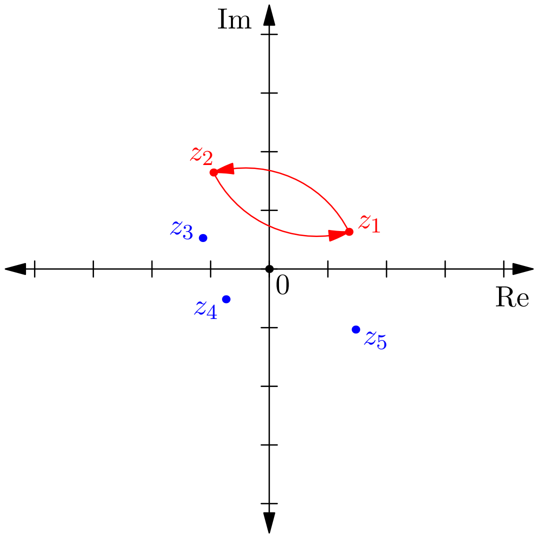
> **Question 34.1.2.** Convince yourself any F indeed must track which root is which when moving roots along smooth paths.

> **Remark 34.1.3** — This isn't true if we bring even more complicated functions such as **Bring Radicals** to the table. But this wasn't really considered "fair game."

§34.2 Step 1: The Simplest Case

Let's first ignore the $\sqrt[n]{}$ operator for motivation. Suppose I told you that some rational function R always finds a root of a quintic polynomial $P(z) = (z - z_1)(z - z_2)(z - z_3)(z - z_4)(z - z_5)$. For simplicity, let all the roots be distinct.

Suppose that initially R outputs z_1. Consider what happens we smoothly swap the roots z_1 and z_2 along two non-intersecting paths that doesn't go through other roots.



Since R is continuous, it must be tracking the same root. However, once we finish swapping z_1 and z_2, the coefficients of P are the same as they were initially. But this means that R has been tricked into changing the root it outputs, contradiction!

The bigger picture here is that we were able to find an operation that fixes R while changing the order of the roots in S_5.

§34.3 Step 2: Nested Roots

Once we add $\sqrt[n]{}$ back to the picture, this idea no longer works right out of the box.

Example 34.3.1 (Quadratic Formula)

If you've done any competition math, you know that for a polynomial $P(z) = az^2 + bz + c = (z - z_1)(z - z_2)$, it follows that the two branches of

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

give z_1 and z_2.

So why can't swapping z_1 and z_2 yield a contradiction here? It's because while all the coefficients end up in the starting position, the liftings of how $\sqrt{b^2 - 4ac}$ travels may not.

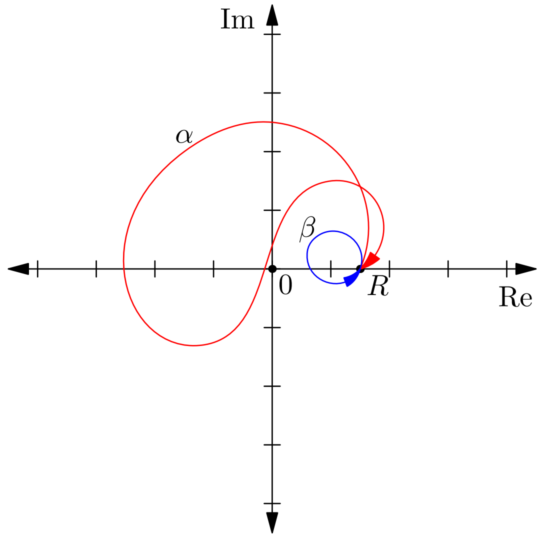
Exercise 34.3.2. Consider the polynomial $z^2 - 1$. Then smoothly swap the roots to get the intermediary polynomials of $(z - e^{it})(z + e^{it})$. See that the two roots given by the quadratic formula also swap position.

Let's now consider the next simplest case of the nth root of a rational function $\sqrt[n]{R}$, and try to fix it with a nontrivial permutation of the roots.

Swapping the roots z_1 and z_2, we keep R the same, but R's path α around the origin may have accumulated some change in phase $2\pi a$. If we were to unswap z_1 and z_2 in the same manner, we'd undo the change in phase, but we'd also be back to doing nothing.

However, while changes in phase are *abelian*, permutations are not. Let's consider another operation of swapping the roots z_2 and z_3. Taking a commutator of the two operations, we keep all the phases the same, but end up with a permutation $(12)(23)(12)^{-1}(23)^{-1}$.

If we mark the second operation's path with β, this corresponds to $\alpha\beta\alpha^{-1}\beta^{-1}$.



Exercise 34.3.3. Show that this permutation operation is nontrivial.

We now have better tools: We have permutations in S_5 that fix the nth roots of rational functions, and their compositions under $+$, $-$, $\times$, $\div$.

How do we handle the nested radicals now?

Example 34.3.4 (Cubic Formula)

The cubic formula contains a nasty term

$$\sqrt[3]{\frac{2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^d)^2 - 4(b^3 - 3ac)^3}}{2}}.$$

Here, we've taken multiple roots.

Definition 34.3.5. Define the degree of a nested radical as the maximum number of times radicals can be found in other radicals.

Let's now consider nested radicals of degree 2, such as say $\sqrt[3]{\sqrt{ab+c} - \sqrt{d}}$. We know that we have nontrivial commutators σ and ρ that fix the interior of the cube root, but once again the phase may not be preserved under each operation individually. Once again, we can again consider the *commutators* of these commutators, say $\sigma\rho\sigma^{-1}\rho^{-1}$ which by the same logic fixes the issues with phase.

There's no reason, we can't consider the commutators of commutators of commutators to fix radicals of degree 3 and so on. It thus just remains that we always can keep getting nontrivial commutators.

§34.4 Step 3: Normal Groups

We've reduced this to a group theory problem. Given a chain of commutators

$$S_5 = G \supseteq G^{(1)} \supseteq G^{(2)} \supseteq \dots$$

where each group is the commutator subgroup of the next, we want to show that $G^{(n)}$ never becomes trivial. This chain is called the **derived series**.

> **Exercise 34.4.1.** Show that for the commutator subgroup $[G, G]$ of a group G, we have that $[G, G] \trianglelefteq G$, and that $G/[G, G]$ is Abelian.

Definition 34.4.2. A group G is **solvable** if its derived series is nontrivial.

So all that remains is showing that S_5 is not solvable. This is a calculation that isn't relevant to the topology ideas in this chapter, so we defer it to Problem 34A.

§34.5 Summary

While this is indeed a valid proof, it has some pros and cons. As a con, we haven't shown that any polynomial such as $z^5 - z - 1$ has a root that can't be expressed using nested nth roots. We've only that we don't have a formula for all degree 5 polynomials.

As a pro, this argument makes it easy to add even more functions such as exp, sin, and cos to the mix and show even then that no such formula exists. It also allows you to broadly understand what people mean when they compare this theorem to a fact that A_5 is not solvable.

§34.6 A few harder problems to think about

Problem 34A. Show that A_5 is not solvable.