VIII

Calculus 101

Part VIII: Contents

26 Limits and series

Now that we have developed the theory of metric (and topological) spaces well, we give a three-chapter sequence which briskly covers the theory of single-variable calculus.

Much of the work has secretly already been done, For example, if x_n and y_n are real sequences with $\lim_n x_n = x$ and $\lim_n y_n = y$, then in fact $\lim_n(x_n + y_n) = x + y$ or $\lim_n(x_n y_n) = xy$, because we showed in Proposition 2.5.5 that arithmetic was continuous. We will also see that completeness plays a crucial role.

§26.1 Completeness and inf/sup

Prototypical example for this section: $\sup[0, 1] = \sup(0, 1) = 1$.

As $\mathbb{R}$ is a metric space, we may discuss continuity and convergence. There are two important facts about $\mathbb{R}$ which will make most of the following sections tick.

The first fact you have already seen before:

Theorem 26.1.1 ($\mathbb{R}$ is complete)

As a metric space, $\mathbb{R}$ is complete: sequences converge if and only if they are Cauchy.

The second one we have not seen before — it is the existence of inf and sup. Your intuition should be:

sup is max adjusted slightly for infinite sets. (And inf is adjusted min.)

Why the "adjustment"?

Example 26.1.2 (Why is max not good enough?)

Let's say we have the open interval $S = (0, 1)$. The elements can get arbitrarily close to 1, so we would like to think "1 is the max of S"; except the issue is that $1 \notin S$. In general, infinite sets don't necessarily *have* a maximum, and we have to talk about bounds instead.

So we will define $\sup S$ in such a way that $\sup S = 1$. The definition is that "1 is the smallest number which is at least every element of S".

To write it out:

Definition 26.1.3. If S is a set of real numbers:

- An *upper bound* for S is a real number M such that $x \le M$ for all $x \in S$. If one exists, we say S is **bounded above**;
- A *lower bound* for S is a real number m such that $m \le x$ for all $x \in S$. If one exists, we say S is **bounded below**.
- If both upper and lower bounds exist, we say S is **bounded**.

Theorem 26.1.4 ($\mathbb{R}$ has inf's and sup's)
Let S be a nonempty set of real numbers.

- If S is bounded above then it has a *least* upper bound, which we denote by $\sup S$ and refer to as the **supremum** of S.
- If S is bounded below then it has a *greatest* lower bound, which we denote by $\inf S$ and refer to as the **infimum** of S.

Definition 26.1.5. For convenience, if S has not bounded above, we write $\sup S = +\infty$. Similarly, if S has not bounded below, we write $\inf S = -\infty$.

Example 26.1.6 (Supremums)
Since the examples for infimums are basically the same, we stick with supremums for now.

(a) If $S = \{1, 2, 3, \dots\}$ then S is not bounded above, so we have $\sup S = +\infty$.

(b) If $S = \{\dots, -2, -1\}$ denotes the set of negative integers, then $\sup S = -1$.

(c) Let $S = [0, 1]$ be a closed interval. Then $\sup S = 1$.

(d) Let $S = (0, 1)$ be an open interval. Then $\sup S = 1$ as well, even though 1 itself is not an element of S.

(e) Let $S = \mathbb{Q} \cap (0, 1)$ denote the set of rational numbers between 0 and 1. Then $\sup S = 1$ still.

(f) If S is a finite nonempty set, then $\sup S = \max S$.

Definition 26.1.7 (Porting definitions to sequences)**.** If $a_1, \dots$ is a sequence we will often write

$$\sup_n a_n := \sup \{a_n \mid n \in \mathbb{N}\}$$
$$\inf_n a_n := \inf \{a_n \mid n \in \mathbb{N}\}$$

for the supremum and infimum of the set of elements of the sequence. We also use the words "bounded above/below" for sequences in the same way.

Example 26.1.8 (Infimum of a sequence)
The sequence $a_n = \frac{1}{n}$ has infimum $\inf a_n = 0$.

§26.2 Proofs of the two key completeness properties of $\mathbb{R}$

Careful readers will note that we have not actually proven either Theorem 26.1.4 or Theorem 26.1.1. We will do so here.

First, we show that the ability to take infimums and supremums lets you prove completeness of $\mathbb{R}$.

Proof that Theorem 26.1.4 implies Theorem 26.1.1. Let $a_1, a_2, \dots$ be a Cauchy sequence. By discarding finitely many leading terms, we may as well assume that $|a_i - a_j| \le 100$ for all i and j. In particular, the sequence is now bounded; it lies between $[a_1 - 100, a_1 + 100]$ for example.

We want to show this sequence converges, so we have to first describe what the limit is. We know that to do this we are really going to have to use the fact that we live in $\mathbb{R}$. (For example we know in $\mathbb{Q}$ the limit of 1, 1.4, 1.41, 1.414, ... is nonexistent.)

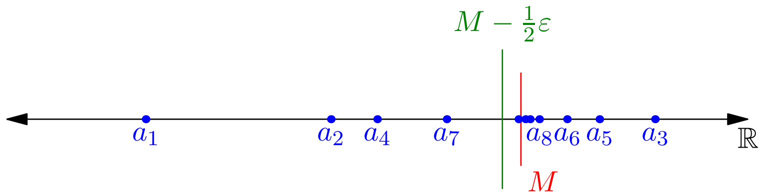
We propose the following: let

$$S = \{x \in \mathbb{R} \mid a_n \ge x \text{ for infinitely many } n\}.$$

We claim that the sequence converges to $M = \sup S$.

> **Exercise 26.2.1.** Show that this supremum makes sense by proving that $a_1 - 100 \in S$ (so S is nonempty) while all elements of S are at most $a_1 + 100$ (so S is bounded above). Thus we are allowed to actually take the supremum.

You can think of this set S with the following picture. We have a Cauchy sequence drawn in the real line which we think converges, which we can visualize as a bunch of dots on the real line, with some order on them. We wish to cut the line with a knife such that only finitely many dots are to the left of the knife. (For example, placing the knife all the way to the left always works.) The set S represents the places where we could put the knife, and M is "as far right" as we could go. Because of the way supremums work, M might not *itself* be a valid knife location, but certainly anything to its left is.



Let $\varepsilon > 0$ be given; we want to show eventually all terms are within ε of M. Because the sequence is Cauchy, there is an N such that eventually $|a_m - a_n| < \frac{1}{2}\varepsilon$ for $m \ge n \ge N$.

Now suppose we fix n and vary m. By the definition of M, it should be possible to pick the index m such that $a_m \ge M - \frac{1}{2}\varepsilon$ (there are infinitely many to choose from since $M - \frac{1}{2}\varepsilon$ is a valid knife location, and we only need $m \ge n$). In that case we have

$$|a_n - M| \le |a_n - a_m| + |a_m - M| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

by the triangle inequality. This completes the proof. $\square$

Therefore it is enough to prove the latter Theorem 26.1.4. To do this though, we would need to actually give a rigorous definition of the real numbers $\mathbb{R}$, since we have not done so yet!

One approach that makes this easy is to use the so-called **Dedekind cut** construction. Suppose we take the rational numbers $\mathbb{Q}$. Then one *defines* a real number to be a "cut" $A \mid B$ of the set of rational numbers: a pair of subsets of $\mathbb{Q}$ such that

- $\mathbb{Q} = A \sqcup B$ is a disjoint union;
- A and B are nonempty;
- we have $a < b$ for every $a \in A$ and $b \in B$, and

- A has no largest element (i.e. $\sup A \notin A$).

This can again be visualized by taking what you think of as the real line, and slicing at some real number. The subset $\mathbb{Q} \subset \mathbb{R}$ gets cut into two halves A and B. If the knife happens to land exactly at a rational number, by convention we consider that number to be in the right half (which explains the last fourth condition that $\sup A \notin A$).

With this definition Theorem 26.1.4 is easy: to take the supremum of a set of real numbers, we take the union of all the left halves. The hard part is then figuring out how to define $+$, $-$, $\times$, $\div$ and so on with this rather awkward construction. If you want to read more about this construction in detail, my favorite reference is [Pu02], in which all of this is done carefully in Chapter 1.

§26.3 Monotonic sequences

Here is a great exercise.

> **Exercise 26.3.1** (Mandatory)**.** Prove that if $a_1 \ge a_2 \ge \dots \ge 0$ then the limit
>
> $$\lim_{n\to\infty} a_n$$
>
> exists. Hint: the idea in the proof of the previous section helps; you can also try to use completeness of $\mathbb{R}$. Second hint: if you are really stuck, wait until after Proposition 26.4.5, at which point you can use essentially copy its proof.

The proof here readily adapts by shifting.

Definition 26.3.2. A sequence a_n is **monotonic** if either $a_1 \ge a_2 \ge \dots$ or $a_1 \le a_2 \le \dots$.

Theorem 26.3.3 (Monotonic bounded sequences converge)
Let $a_1, a_2, \dots$ be a monotonic bounded sequence. Then $\lim_{n\to\infty} a_n$ exists.

Example 26.3.4 (Silly example of monotonicity)
Consider the sequence defined by

$$\begin{aligned} a_1 &= 1.2 \\ a_2 &= 1.24 \\ a_3 &= 1.248 \\ a_4 &= 1.24816 \\ a_5 &= 1.2481632 \\ &\vdots \end{aligned}$$

and so on, where in general we stuck on the decimal representation of the next power of 2. This will converge to *some* real number, although of course this number is quite unnatural and there is probably no good description for it.

In general, "infinite decimals" can now be defined as the limit of the truncated finite ones.

Example 26.3.5 ($0.9999\cdots = 1$)

In particular, I can finally make precise the notion you argued about in elementary school that

$$0.9999\cdots = 1.$$

We simply *define* a repeating decimal to be the limit of the sequence 0.9, 0.99, 0.999.... And it is obvious that the limit of this sequence is 1.

Some of you might be a little surprised since it seems like we really should have $0.9999 = 9 \cdot 10^{-1} + 9 \cdot 10^{-2} + \dots$ — the limit of "partial sums". Don't worry, we're about to define those in just a moment.

Here is one other great use of monotonic sequences.

Definition 26.3.6. Let a_1, a_2, ... be a sequence (not necessarily monotonic) which is bounded below. We define

$$\limsup_{n\to\infty} a_n := \lim_{N\to\infty} \sup_{n \ge N} a_n = \lim_{N\to\infty} \sup \{a_N, a_{N+1}, \dots\}.$$

This is called the **limit supremum** of (a_n). We set $\limsup_{n\to\infty} a_n$ to be $+\infty$ if a_n is not bounded above.

If a_n is bounded above, the **limit infimum** $\liminf_{n\to\infty} a_n$ is defined similarly. In particular, $\liminf_{n\to\infty} a_n = -\infty$ if a_n is not bounded below.

Exercise 26.3.7. Show that these definitions make sense, by checking that the supremums are non-increasing, and bounded below.

We can think of $\limsup_n a_n$ as "supremum, but allowing finitely many terms to be discarded".

§26.4 Infinite series

Prototypical example for this section: $\sum_{k\ge 1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1$.

We will actually begin by working with infinite series, since in the previous chapters we defined limits of sequences, and so this is actually the next closest thing to work with.[1]

This will give you a rigorous way to think about statements like

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

and help answer questions like "how can you add rational numbers and get an irrational one?".

[1]Conceptually: discrete things are easier to be rigorous about than continuous things, so series are actually "easier" than derivatives! I suspect the reason that most schools teach series last in calculus is that most calculus courses do not have proofs.

Definition 26.4.1. Consider a sequence $a_1, \dots$ of real numbers. The series $\sum_k a_k$ **converges** to a limit L if the sequence of "partial sums"

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + \dots + a_n \end{aligned}$$

converges to the limit L. Otherwise it **diverges**.

Abuse of Notation 26.4.2 (Writing divergence as $+\infty$)**.** It is customary, if all the a_k are nonnegative, to write $\sum_k a_k = \infty$ to denote that the series diverges.

You will notice that by using the definition of sequences, we have masterfully sidestepped the issue of "adding infinitely many numbers" which would otherwise cause all sorts of problems.

An "infinite sum" is actually the *limit* of its partial sums. There is no infinite addition involved.

That's why it's for example okay to have $\sum_{n\ge1} \frac{1}{n^2} = \frac{\pi^2}{6}$ be irrational; we have already seen many times that sequences of rational numbers can converge to irrational numbers. It also means we can gladly ignore all the irritating posts by middle schoolers about $1 + 2 + 3 + \dots = -\frac{1}{12}$; the partial sums explode to $+\infty$, end of story, and if you want to assign a value to that sum it had better be a definition.

Example 26.4.3 (The classical telescoping series)

We can now prove the classic telescoping series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

in a way that doesn't just hand-wave the ending. Note that the kth partial sum is

$$\begin{aligned} \sum_{k=1}^{n} \frac{1}{k(k+1)} &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)} \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

The limit of this partial sum as $n \to \infty$ is 1.

Example 26.4.4 (Harmonic series diverges)

We can also make sense of the statement that $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$ (i.e. it diverges). We may bound the 2^nth partial sums from below:

$$\begin{aligned}\sum_{k=1}^{2^n} \frac{1}{k} &= \frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{2^n} \\ &\ge \frac{1}{1} + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) \\ &+ \dots + \underbrace{\left(\frac{1}{2^n} + \dots + \frac{1}{2^n}\right)}_{2^{n-1} \text{ terms}} \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2} = 1 + \frac{n-1}{2}.\end{aligned}$$

A sequence satisfying $s_{2^n} \ge 1 + \frac{1}{2}(n-1)$ will never converge to a finite number!

I had better also mention that for nonnegative sums, convergence is just the same as having "finite sum" in the following sense.

Proposition 26.4.5 (Partial sums of nonnegatives bounded implies convergent)
Let $\sum_k a_k$ be a series of *nonnegative* real numbers. Then $\sum_k a_k$ converges to some limit if and only if there is a constant M such that

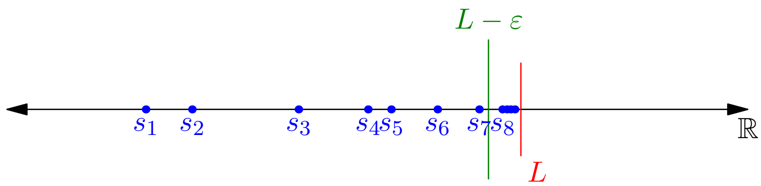
$$a_1 + \dots + a_n < M$$

for every positive integer n.

Proof. This is actually just Theorem 26.3.3 in disguise, but since we left the proof as an exercise back then, we'll write it out this time.

Obviously if no such M exists then convergence will not happen, since this means the sequence s_n of partial sums is unbounded.

Conversely, if such M exists then we have $s_1 \le s_2 \le \dots < M$. Then we contend the sequence s_n converges to $L := \sup_n s_n < \infty$. (If you read the proof that completeness implies Cauchy, the picture is nearly the same here, but simpler.)



Indeed, this means for any ε there are infinitely many terms of the sequence exceeding $L - \varepsilon$; but since the sequence is monotonic, once $s_n \ge L - \varepsilon$ then $s_{n'} \ge L - \varepsilon$ for all $n' \ge n$. This implies convergence. $\square$

Abuse of Notation 26.4.6 (Writing $\sum < \infty$)**.** For this reason, if a_k are nonnegative real numbers, it is customary to write

$$\sum_k a_k < \infty$$

as a shorthand for "$\sum_k a_k$ converges to a finite limit", (or perhaps shorthand for "$\sum_k a_k$ is bounded" — as we have just proved these are equivalent). We will use this notation too.

§26.5 Series addition is not commutative: a horror story

One unfortunate property of the above definition is that it actually depends on the order of the elements. In fact, it turns out that there is an explicit way to describe when rearrangement is okay.

Definition 26.5.1. A series $\sum_k a_k$ of real numbers is said to **converge absolutely** if

$$\sum_k |a_k| < \infty$$

i.e. the series of absolute values converges to some limit. If the series converges, but not absolutely, we say it **converges conditionally**.

Proposition 26.5.2 (Absolute convergence $\implies$ convergence)

If a series $\sum_k a_k$ of real numbers converges absolutely, then it converges in the usual sense.

Exercise 26.5.3 (Great exercise)**.** Prove this by using the Cauchy criteria: show that if the partial sums of $\sum_k |a_k|$ are Cauchy, then so are the partial sums of $\sum_k a_k$.

Then, rearrangement works great.

Theorem 26.5.4 (Permutation of terms okay for absolute convergence)

Consider a series $\sum_k a_k$ which is absolutely convergent and has limit L. Then any permutation of the terms will also converge to L.

Proof. Suppose $\sum_k a_k$ converges to L, and b_n is a rearrangement. Let $\varepsilon > 0$. We will show that the partial sums of b_n are eventually within ε of L.

The hypothesis means that there is a large N in terms of ε such that

$$\left| \sum_{k=1}^{N} a_k - L \right| < \frac{1}{2}\varepsilon \quad \text{and} \quad \sum_{k=N+1}^{n} |a_k| < \frac{1}{2}\varepsilon$$

for every $n \geq N$ (the former from vanilla convergence of a_k and the latter from the fact that a_k converges absolutely, hence its partial sums are Cauchy).

Now suppose M is large enough that $a_1, \dots, a_N$ are contained within the terms $\{b_1, \dots, b_M\}$. Then

$$\begin{aligned} b_1 + \dots + b_M = &(a_1 + \dots + a_N) \\ &+ \underbrace{a_{i_1} + a_{i_2} + \dots + a_{i_{M-N}}}_{M-N \text{ terms with indices } > N} \end{aligned}$$

The terms in the first line sum up to within $\frac{1}{2}\varepsilon$ of L, and the terms in the second line have sum at most $\frac{1}{2}\varepsilon$ in absolute value, so the total $b_1 + \dots + b_M$ is within $\frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$ of L. □

In particular, when you have nonnegative terms, the world is great:

Nonnegative series can be rearranged at will.

And the good news is that actually, in practice, most of your sums will be nonnegative.

The converse is not true, and in fact, it is almost the worst possible converse you can imagine.

Theorem 26.5.5 (Riemann rearrangement theorem: Permutation of terms meaningless for conditional convergence)

Consider a series $\sum_k a_k$ which converges *conditionally* to some real number. Then, there exists a permutation of the series which converges conditionally to 1337.

(Or any constant. You can also get it to diverge, too.)

So, permutation is as bad as possible for conditionally convergent series, and hence don't even bother to try.

§26.6 Limits of functions at points

Prototypical example for this section: $\lim_{x\to\infty} 1/x = 0$.

We had also better define the notion of a limit of a real function, which (surprisingly) we haven't actually defined yet. The definition will look like what we have seen before with continuity.

Definition 26.6.1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function[2] and let $p \in \mathbb{R}$ be a point in the domain. Suppose there exists a real number L such that:

> For every $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x - p| < \delta$ and $x \neq p$ then $|f(x) - L| < \varepsilon$.

Then we say L is the **limit** of f as $x \to p$, and write

$$\lim_{x\to p} f(x) = L.$$

There is an important point here: in this definition we *deliberately* require that $x \neq p$.

The value $\lim_{x\to p} f(x)$ does not depend on $f(p)$, and accordingly we often do not even bother to define $f(p)$.

Example 26.6.2 (Function with a hole)

Define the function $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 3x & \text{if } x \neq 0 \\ 2019 & \text{otherwise.} \end{cases}$$

Then $\lim_{x\to 0} f(x) = 0$. The value $f(0) = 2019$ does not affect the limit. Obviously, because $f(0)$ was made up to be some artificial value that did not agree with the limit, this function is discontinuous at $x = 0$.

[2] Or $f\colon (a,b) \to \mathbb{R}$, or variants. We just need f to be defined on an open neighborhood of p.

Question 26.6.3 (Mandatory)**.** Show that a function f is continuous at p if and only if $\lim_{x \to p} f(x)$ exists and equals $f(p)$.

Example 26.6.4 (Less trivial example: a rational piecewise function)
Define the function $f \colon \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } q > 0 \text{ and } \gcd(p,q) = 1 \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

For example, $f(\pi) = 0$, $f(2/3) = \frac{1}{3}$, $f(0.17) = \frac{1}{100}$. Then

$$\lim_{x \to 0} f(x) = 0.$$

For example, if $|x| < 1/100$ and $x \neq 0$ then $f(x)$ is either zero (for x irrational) or else is at most $\frac{1}{101}$ (if x is rational).

As $f(0) = 1$, this function is also discontinuous at $x = 0$. However, if we change the definition so that $f(0) = 0$ instead, then f becomes continuous at 0.

Example 26.6.5 (Famous example)
Let $f(x) = \frac{\sin x}{x}$, $f \colon \mathbb{R} \to \mathbb{R}$, where $f(0)$ is assigned any value. Then

$$\lim_{x \to 0} f(x) = 1.$$

We will not prove this here, since I don't want to get into trig yet. In general, I will basically only use trig functions for examples and not for any theory, so most properties of the trig functions will just be quoted.

Abuse of Notation 26.6.6 (The usual notation)**.** From now on, the above example will usually be abbreviated to just

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

The reason there is a slight abuse here is that I'm supposed to feed a function f into the limit, and instead I've written down an expression which is defined everywhere — except at $x = 0$. But that $f(0)$ value doesn't change anything. So the above means: "the limit of the function described by $f(x) = \frac{\sin x}{x}$, except $f(0)$ can be whatever it wants because it doesn't matter".

Remark 26.6.7 (For metric spaces) — You might be surprised that I didn't define the notion of $\lim_{x \to p} f(x)$ earlier for $f \colon M \to N$ a function on metric spaces. We can actually do so as above, but there is one nuance: what if our metric space M is discrete, so p has no points nearby it? (Or even more simply, what if M is a one-point space?) We then cannot define $\lim_{x \to p} f(x)$ at all.

Thus if $f \colon M \to N$ and we want to define $\lim_{x \to p} f(x)$, we have the requirement that p should have a point within ε of it, for any $\varepsilon > 0$. In other words, p should not be an isolated point.

As usual, there are no surprises with arithmetic, we have $\lim_{x\to p}(f(x) \pm g(x)) = \lim_{x\to p} f(x) \pm \lim_{x\to p} g(x)$, and so on and so forth. We have effectively done this proof before so we won't repeat it again.

§26.7 Limits of functions at infinity

Annoyingly, we actually have to make this definition separately, even though it will not feel any different from earlier examples.

Definition 26.7.1. Let $f: \mathbb{R} \to \mathbb{R}$. Suppose there exists a real number L such that:

> For every $\varepsilon > 0$, there exists a constant M such that if $x > M$, then $|f(x) - L| < \varepsilon$.

Then we say L is the **limit** of f as x approaches ∞ and write

$$\lim_{x\to\infty} f(x) = L.$$

The limit $\lim_{x\to-\infty} f(x)$ is defined similarly, with $x > M$ replaced by $x < M$.

Fortunately, as ∞ is not an element of $\mathbb{R}$, we don't have to do the same antics about $f(\infty)$ like we had to do with "$f(p)$ set arbitrarily". So these examples can be more easily written down.

Example 26.7.2 (Limit at infinity)
The usual:
$$\lim_{x\to\infty} \frac{1}{x} = 0.$$
I'll even write out the proof: for any $\varepsilon > 0$, if $x > 1/\varepsilon$ then $\left|\frac{1}{x} - 0\right| < \varepsilon$.

There are no surprises with arithmetic: we have $\lim_{x\to\infty}(f(x) \pm g(x)) = \lim_{x\to\infty} f(x) \pm \lim_{x\to\infty} g(x)$, and so on and so forth. This is about the fourth time I've mentioned this, so I will not say more.

§26.8 A few harder problems to think about

Problem 26A. Define the sequence

$$a_n = (-1)^n + \frac{n^3}{2^n}$$

for every positive integer n. Compute the limit infimum and the limit supremum.

Problem 26B. For which bounded sequences a_n does $\liminf_n a_n = \limsup_n a_n$?

Problem 26C† (Comparison test). Let $\sum a_n$ and $\sum b_n$ be two series. Assume $\sum b_n$ is absolutely convergent, and $|a_n| \le |b_n|$ for all integers n. Prove that $\sum_n a_n$ is absolutely convergent.

Problem 26D (Geometric series)**.** Let $-1 < r < 1$ be a real number. Show that the series

$$1 + r + r^2 + r^3 + \dots$$

converges absolutely and determine what it converges to.

Problem 26E (Alternating series test)**.** Let $a_0 \ge a_1 \ge a_2 \ge a_3 \ge \dots$ be a weakly decreasing sequence of nonnegative real numbers, and assume that $\lim_{n\to\infty} a_n = 0$. Show that the series $\sum_n (-1)^n a_n$ is convergent (it need not be absolutely convergent).

Problem 26F ([Pu02, Chapter 3, Exercise 55])**.** Let $(a_n)_{n\ge1}$ and $(b_n)_{n\ge1}$ be sequences of real numbers. Assume $a_1 \le a_2 \le \dots \le 1000$ and moreover that $\sum_n b_n$ converges. Prove that $\sum_n a_n b_n$ converges. (Note that in both the hypothesis and statement, we do not have absolute convergence.)

Problem 26G (Putnam 2016 B1)**.** Let $x_0, x_1, x_2, \dots$ be the sequence such that $x_0 = 1$ and for $n \ge 0$,

$$x_{n+1} = \log(e^{x_n} - x_n)$$

(as usual, log is the natural logarithm). Prove that the infinite series $x_0 + x_1 + \dots$ converges and determine its value.

Problem 26H. Consider again the function $f\colon \mathbb{R} \to \mathbb{R}$ in Example 26.6.4 defined by

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } q > 0 \text{ and } \gcd(p,q) = 1 \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

For every real number p, compute $\lim_{x\to p} f(x)$, if it exists. At which points is f continuous?

27 Bonus: A hint of p-adic numbers

This is a bonus chapter meant for those who have also read about **rings and fields**: it's a nice tidbit at the intersection of algebra and analysis.

In this chapter, we are going to redo most of the previous chapter with the absolute value $|-|$ replaced by the p-adic one. This will give us the p-adic integers $\mathbb{Z}_p$, and the p-adic numbers $\mathbb{Q}_p$. The one-sentence description is that these are "integers/rationals carrying full mod p^e information" (and only that information).

In everything that follows p is always assumed to denote a prime. The first four sections will cover the founding definitions culminating in a short solution to a USA TST problem. We will then state (mostly without proof) some more surprising results about continuous functions $f \colon \mathbb{Z}_p \to \mathbb{Q}_p$; finally we close with the famous proof of the Skolem-Mahler-Lech theorem using p-adic analysis.

§27.1 Motivation

Before really telling you what $\mathbb{Z}_p$ and $\mathbb{Q}_p$ are, let me tell you what you might expect them to do.

In elementary/olympiad number theory, we're already well-familiar with the following two ideas:

- Taking modulo a prime p or prime power p^e, and
- Looking at the exponent ν_p.

Let me expand on the first point. Suppose we have some Diophantine equation. In olympiad contexts, one can take an equation modulo p to gain something else to work with. Unfortunately, taking modulo p loses some information: the reduction $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p$ is far from injective.

If we want finer control, we could consider instead taking modulo p^2, rather than taking modulo p. This can also give some new information (cubes modulo 9, anyone?), but it has the disadvantage that $\mathbb{Z}/p^2$ isn't a field, so we lose a lot of the nice algebraic properties that we got if we take modulo p.

One of the goals of p-adic numbers is that we can get around these two issues I described. The p-adic numbers we introduce is going to have the following properties:

1. **You can "take modulo p^e for all e at once".** In olympiad contexts, we are used to picking a particular modulus and then seeing what happens if we take that modulus. But with p-adic numbers, we won't have to make that choice. An equation of p-adic numbers carries enough information to take modulo p^e.
2. **The numbers $\mathbb{Q}_p$ form a field**, the nicest possible algebraic structure: $1/p$ makes sense. Contrast this with $\mathbb{Z}/p^2$, which is not even an integral domain.
3. **It doesn't lose as much information** as taking modulo p does: rather than the surjective $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p$ we have an *injective* map $\mathbb{Z} \hookrightarrow \mathbb{Z}_p$.

4. **Despite this, you "ignore" some "irrelevant" data**. Just like taking modulo p, you want to zoom-in on a particular type of algebraic information, and this means necessarily losing sight of other things.[1]

So, you can think of p-adic numbers as the right tool to use if you only really care about modulo p^e information, but normal $\mathbb{Z}/p^e$ isn't quite powerful enough.

To be more concrete, I'll give a poster example now:

Example 27.1.1 (USA TST 2002/2)

For a prime p, show the value of

$$f_p(x) = \sum_{k=1}^{p-1} \frac{1}{(px+k)^2} \pmod{p^3}$$

does not depend on x.

Here is a problem where we *clearly* only care about p^e-type information. Yet it's a nontrivial challenge to do the necessary manipulations mod p^3 (try it!). The basic issue is that there is no good way to deal with the denominators modulo p^3 (in part $\mathbb{Z}/p^3$ is not even an integral domain).

However, with p-adic analysis we're going to be able to overcome these limitations and give a "straightforward" proof by using the identity

$$\left(1 + \frac{px}{k}\right)^{-2} = \sum_{n \ge 0} \binom{-2}{n} \left(\frac{px}{k}\right)^n.$$

Such an identity makes no sense over $\mathbb{Q}$ or $\mathbb{R}$ for convergence reasons, but it will work fine over $\mathbb{Q}_p$, which is all we need.

§27.2 Algebraic perspective

Prototypical example for this section: $-1/2 = 1 + 3 + 3^2 + 3^3 + \cdots \in \mathbb{Z}_3$.

We now construct $\mathbb{Z}_p$ and $\mathbb{Q}_p$. I promised earlier that a p-adic integer will let you look at "all residues modulo p^e" at once. This definition will formalize this.

§27.2.i Definition of $\mathbb{Z}_p$

Definition 27.2.1 (Introducing $\mathbb{Z}_p$)**.** A ***p*-adic integer** is a sequence

$$x = (x_1 \bmod p,\ x_2 \bmod p^2,\ x_3 \bmod p^3,\ \dots)$$

of residues x_e modulo p^e for each integer e, satisfying the compatibility relations $x_i \equiv x_j \pmod{p^i}$ for $i < j$.

The set $\mathbb{Z}_p$ of p-adic integers forms a ring under component-wise addition and multiplication.

[1] To draw an analogy: the equation $a^2+b^2+c^2+d^2 = -1$ has no integer solutions, because, well, squares are nonnegative. But you will find that this equation has solutions modulo any prime p, because once you take modulo p you stop being able to talk about numbers being nonnegative. The same thing will happen if we work in p-adics: the above equation has a solution in $\mathbb{Z}_p$ for every prime p.

Example 27.2.2 (Some 3-adic integers)

Let $p = 3$. Every usual integer n generates a (compatible) sequence of residues modulo p^e for each e, so we can view each ordinary integer as p-adic one:

$$50 = (2 \bmod 3,\ 5 \bmod 9,\ 23 \bmod 27,\ 50 \bmod 81,\ 50 \bmod 243,\ \dots).$$

On the other hand, there are sequences of residues which do not correspond to any usual integer despite satisfying compatibility relations, such as

$$(1 \bmod 3,\ 4 \bmod 9,\ 13 \bmod 27,\ 40 \bmod 81,\ \dots)$$

which can be thought of as $x = 1 + p + p^2 + \dots$.

In this way we get an injective map

$$\mathbb{Z} \hookrightarrow \mathbb{Z}_p \qquad n \mapsto \left(n \bmod p, n \bmod p^2, n \bmod p^3, \dots\right)$$

which is not surjective. So there are more p-adic integers than usual integers.

(Remark for experts: those of you familiar with category theory might recognize that this definition can be written concisely as

$$\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^e\mathbb{Z}$$

where the inverse limit is taken across $e \ge 1$.)

Exercise 27.2.3. Check that $\mathbb{Z}_p$ is an integral domain.

§27.2.ii Base p expansion

Here is another way to think about p-adic integers using "base p". As in the example earlier, every usual integer can be written in base p, for example

$$50 = \overline{1212}_3 = 2 \cdot 3^0 + 1 \cdot 3^1 + 2 \cdot 3^2 + 1 \cdot 3^3.$$

More generally, given any $x = (x_1, \dots) \in \mathbb{Z}_p$, we can write down a "base p" expansion in the sense that there are exactly p choices of x_k given x_{k-1}. Continuing the example earlier, we would write

$$\begin{aligned}(1 \bmod 3,\ 4 \bmod 9,\ 13 \bmod 27,\ 40 \bmod 81,\ \dots) &= 1 + 3 + 3^2 + \dots \\ &= \overline{\dots 1111}_3\end{aligned}$$

and in general we can write

$$x = \sum_{k \ge 0} a_k p^k = \overline{\dots a_2 a_1 a_0}_p$$

where $a_k \in \{0, \dots, p-1\}$, such that the equation holds modulo p^e for each e. Note the expansion is infinite to the *left*, which is different from what you're used to.

(Amusingly, negative integers also have infinite base p expansions: $-4 = \overline{\dots 222212}_3$, corresponding to $(2 \bmod 3,\ 5 \bmod 9,\ 23 \bmod 27,\ 77 \bmod 81 \dots)$.)

Thus you may often hear the advertisement that a p-adic integer is a "possibly infinite base p expansion". This is correct, but later on we'll be thinking of $\mathbb{Z}_p$ in a more and more "analytic" way, and so I prefer to think of this as

p-adic integers are Taylor series with base p.

Indeed, much of your intuition from generating functions $K[[X]]$ (where K is a field) will carry over to $\mathbb{Z}_p$.

§27.2.iii Constructing $\mathbb{Q}_p$

Here is one way in which your intuition from generating functions carries over:

Proposition 27.2.4 (Non-multiples of p are all invertible)
The number $x \in \mathbb{Z}_p$ is invertible if and only if $x_1 \neq 0$. In symbols,

$$x \in \mathbb{Z}_p^\times \iff x \not\equiv 0 \pmod p.$$

Contrast this with the corresponding statement for $K[[X]]$: a generating function $F \in K[[X]]$ is invertible iff $F(0) \neq 0$.

Proof. If $x \equiv 0 \pmod p$ then $x_1 = 0$, so clearly not invertible. Otherwise, $x_e \not\equiv 0 \pmod p$ for all e, so we can take an inverse y_e modulo p^e, with $x_e y_e \equiv 1 \pmod{p^e}$. As the y_e are themselves compatible, the element $(y_1, y_2, \dots)$ is an inverse. □

Example 27.2.5 (We have $-\frac{1}{2} = \overline{\dots 1111}_3 \in \mathbb{Z}_3$)
We claim the earlier example is actually

$$\begin{aligned} -\frac{1}{2} &= (1 \bmod 3,\ 4 \bmod 9,\ 13 \bmod 27,\ 40 \bmod 81,\ \dots) = 1 + 3 + 3^2 + \dots \\ &= \overline{\dots 1111}_3. \end{aligned}$$

Indeed, multiplying it by -2 gives

$$(-2 \bmod 3,\ -8 \bmod 9,\ -26 \bmod 27,\ -80 \bmod 81,\ \dots) = 1.$$

(Compare this with the "geometric series" $1 + 3 + 3^2 + \dots = \frac{1}{1-3}$. We'll actually be able to formalize this later, but not yet.)

Remark 27.2.6 ($\frac{1}{2}$ is an integer for $p > 2$) — The earlier proposition implies that $\frac{1}{2} \in \mathbb{Z}_3$ (among other things); your intuition about what is an "integer" is different here! In olympiad terms, we already knew $\frac{1}{2} \pmod 3$ made sense, which is why calling $\frac{1}{2}$ an "integer" in the 3-adics is correct, even though it doesn't correspond to any element of $\mathbb{Z}$.

Exercise 27.2.7 (Unimportant but tricky)**.** Rational numbers correspond exactly to eventually periodic base p expansions.

With this observation, here is now the definition of $\mathbb{Q}_p$.

Definition 27.2.8 (Introducing $\mathbb{Q}_p$)**.** Since $\mathbb{Z}_p$ is an integral domain, we let $\mathbb{Q}_p$ denote its field of fractions. These are the **p-adic numbers**.

Continuing our generating functions analogy:

$$\mathbb{Z}_p \text{ is to } \mathbb{Q}_p \quad \text{as} \quad K[[X]] \text{ is to } K((X)).$$

This means

$\mathbb{Q}_p$ can be thought of as Laurent series with base p.

and in particular according to the earlier proposition we deduce:

Proposition 27.2.9 ($\mathbb{Q}_p$ looks like formal Laurent series)
Every nonzero element of $\mathbb{Q}_p$ is uniquely of the form

$$p^k u \qquad \text{where } k \in \mathbb{Z},\ u \in \mathbb{Z}_p^\times.$$

Thus, continuing our base p analogy, elements of $\mathbb{Q}_p$ are in bijection with "Laurent series"

$$\sum_{k \ge -n} a_k p^k = \overline{\dots a_2 a_1 a_0 . a_{-1} a_{-2} \dots a_{-n}}_p$$

for $a_k \in \{0, \dots, p-1\}$. So the base p representations of elements of $\mathbb{Q}_p$ can be thought of as the same as usual, but extending infinitely far to the left (rather than to the right).

Remark 27.2.10 (Warning) — The field $\mathbb{Q}_p$ has characteristic *zero*, not p.

Remark 27.2.11 (Warning on fraction field) — This result implies that you shouldn't think about elements of $\mathbb{Q}_p$ as x/y (for $x, y \in \mathbb{Z}_p$) in practice, even though this is the official definition (and what you'd expect from the name $\mathbb{Q}_p$). The only denominators you need are powers of p.

To keep pushing the formal Laurent series analogy, $K((X))$ is usually not thought of as quotient of generating functions but rather as "formal series with some negative exponents". You should apply the same intuition on $\mathbb{Q}_p$.

Remark 27.2.12 — At this point I want to make a remark about the fact $1/p \in \mathbb{Q}_p$, connecting it to the wish-list of properties I had before. In elementary number theory you can take equations modulo p, but if you do the quantity $n/p \bmod p$ doesn't make sense unless you know $n \bmod p^2$. You can't fix this by just taking modulo p^2 since then you need $n \bmod p^3$ to get $n/p \bmod p^2$, ad infinitum. You can work around issues like this, but the nice feature of $\mathbb{Z}_p$ and $\mathbb{Q}_p$ is that you have modulo p^e information for "all e at once": the information of $x \in \mathbb{Q}_p$ packages all the modulo p^e information simultaneously. So you can divide by p with no repercussions.

§27.3 Analytic perspective

§27.3.i Definition

Up until now we've been thinking about things mostly algebraically, but moving forward it will be helpful to start using the language of analysis. Usually, two real numbers are considered "close" if they are close on the number of line, but for p-adic purposes we only care about modulo p^e information. So, we'll instead think of two elements of $\mathbb{Z}_p$ or $\mathbb{Q}_p$ as "close" if they differ by a large multiple of p^e.

For this we'll borrow the familiar ν_p from elementary number theory.

Definition 27.3.1 (p-adic valuation and absolute value)**.** We define the ***p*-adic valuation** $\nu_p \colon \mathbb{Q}_p^\times \to \mathbb{Z}$ in the following two equivalent ways:

- For $x = (x_1, x_2, \dots) \in \mathbb{Z}_p$ we let $\nu_p(x)$ be the largest e such that $x_e \equiv 0 \pmod{p^e}$ (or $e = 0$ if $x \in \mathbb{Z}_p^\times$). Then extend to all of $\mathbb{Q}_p^\times$ by $\nu_p(xy) = \nu_p(x) + \nu_p(y)$.
- Each $x \in \mathbb{Q}_p^\times$ can be written uniquely as $p^k u$ for $u \in \mathbb{Z}_p^\times$, $k \in \mathbb{Z}$. We let $\nu_p(x) = k$.

By convention we set $\nu_p(0) = +\infty$. Finally, define the ***p*-adic absolute value** $\left|\bullet\right|_p$ by

$$\left|x\right|_p = p^{-\nu_p(x)}.$$

In particular $\left|0\right|_p = 0$.

This fulfills the promise that x and y are close if they look the same modulo p^e for large e; in that case $\nu_p(x-y)$ is large and accordingly $\left|x-y\right|_p$ is small.

§27.3.ii Ultrametric space

In this way, $\mathbb{Q}_p$ and $\mathbb{Z}_p$ becomes a metric space with metric given by $\left|x-y\right|_p$.

> **Exercise 27.3.2.** Suppose $f \colon \mathbb{Z}_p \to \mathbb{Q}_p$ is continuous and $f(n) = (-1)^n$ for every $n \in \mathbb{Z}_{\ge 0}$. Prove that $p = 2$.

In fact, these spaces satisfy a stronger form of the triangle inequality than you are used to from $\mathbb{R}$.

Proposition 27.3.3 ($\left|\bullet\right|_p$ is an ultrametric)
For any $x, y \in \mathbb{Z}_p$, we have the **strong triangle inequality**

$$\left|x+y\right|_p \le \max\left\{\left|x\right|_p, \left|y\right|_p\right\}.$$

Equality holds if (but not only if) $\left|x\right|_p \ne \left|y\right|_p$.

However, $\mathbb{Q}_p$ is more than just a metric space: it is a field, with its own addition and multiplication. This means we can do analysis just like in $\mathbb{R}$ or $\mathbb{C}$: basically, any notion such as "continuous function", "convergent series", et cetera has a p-adic analog. In particular, we can define what it means for an infinite sum to converge:

Definition 27.3.4 (Convergence notions)**.** Here are some examples of p-adic analogs of "real-world" notions.

- A sequence $s_1, \dots$ converges to a limit L if $\lim_{n\to\infty} \left|s_n - L\right|_p = 0$.
- The infinite series $\sum_k x_k$ converges if the sequence of partial sums $s_1 = x_1$, $s_2 = x_1 + x_2$, $\dots$, converges to some limit.
- $\dots$ et cetera $\dots$

With this definition in place, the "base p" discussion we had earlier is now true in the analytic sense: if $x = \overline{\dots a_2 a_1 a_0}_p \in \mathbb{Z}_p$ then

$$\sum_{k=0}^{\infty} a_k p^k \quad \text{converges to } x.$$

Indeed, the difference between x and the nth partial sum is divisible by p^n, hence the partial sums approach x as $n \to \infty$.

While the definitions are all the same, there are some changes in properties that should be true. For example, in $\mathbb{Q}_p$ convergence of partial sums is simpler:

Proposition 27.3.5 ($|x_k|_p \to 0$ iff convergence of series)
A series $\sum_{k=1}^{\infty} x_k$ in $\mathbb{Q}_p$ converges to some limit if and only if $\lim_{k\to\infty} |x_k|_p = 0$.

Contrast this with $\sum \frac{1}{n} = \infty$ in $\mathbb{R}$. You can think of this as a consequence of strong triangle inequality.

Proof. By multiplying by a large enough power of p, we may assume $x_k \in \mathbb{Z}_p$. (This isn't actually necessary, but makes the notation nicer.)

Observe that $x_k \pmod p$ must eventually stabilize, since for large enough n we have $|x_n|_p < 1 \iff \nu_p(x_n) \ge 1$. So let a_1 be the eventual residue modulo p of $\sum_{k=0}^{N} x_k \pmod p$ for large N. In the same way let a_2 be the eventual residue modulo p^2, and so on. Then one can check we approach the limit $a = (a_1, a_2, \dots)$. $\square$

§27.3.iii More fun with geometric series

Let's finally state the p-adic analog of the geometric series formula.

Proposition 27.3.6 (Geometric series)
Let $x \in \mathbb{Z}_p$ with $|x|_p < 1$. Then

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots.$$

Proof. Note that the partial sums satisfy $1 + x + x^2 + \dots + x^n = \frac{1-x^n}{1-x}$, and $x^n \to 0$ as $n \to \infty$ since $|x|_p < 1$. $\square$

So, $1 + 3 + 3^2 + \dots = -\frac{1}{2}$ is really a correct convergence in $\mathbb{Z}_3$. And so on.

If you buy the analogy that $\mathbb{Z}_p$ is generating functions with base p, then all the olympiad generating functions you might be used to have p-adic analogs. For example, you can prove more generally that:

Theorem 27.3.7 (Generalized binomial theorem)
If $x \in \mathbb{Z}_p$ and $|x|_p < 1$, then for any $r \in \mathbb{Q}$ we have the series convergence

$$\sum_{n \ge 0} \binom{r}{n} x^n = (1+x)^r.$$

(I haven't defined $(1+x)^r$, but it has the properties you expect.)

§27.3.iv Completeness

Note that the definition of $|\bullet|_p$ could have been given for $\mathbb{Q}$ as well; we didn't need $\mathbb{Q}_p$ to introduce it (after all, we have ν_p in olympiads already). The big important theorem I must state now is:

Theorem 27.3.8 ($\mathbb{Q}_p$ is complete)

The space $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to $|\bullet|_p$.

This is the definition of $\mathbb{Q}_p$ you'll see more frequently; one then defines $\mathbb{Z}_p$ in terms of $\mathbb{Q}_p$ (rather than vice-versa) according to

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : |x|_p \le 1 \right\}.$$

§27.3.v Philosophical notes

Let me justify why this definition is philosophically nice. Suppose you are an ancient Greek mathematician who is given:

Problem for Ancient Greeks. Estimate the value of the sum

$$S = \frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{10000^2}$$

to within 0.001.

The sum S consists entirely of rational numbers, so the problem statement would be fair game for ancient Greece. But it turns out that in order to get a good estimate, it *really helps* if you know about the real numbers: because then you can construct the infinite series $\sum_{n\ge1} n^{-2} = \frac{1}{6}\pi^2$, and deduce that $S \approx \frac{\pi^2}{6}$, up to some small error term from the terms past $\frac{1}{10001^2}$, which can be bounded.

Of course, in order to have access to enough theory to prove that $S = \pi^2/6$, you need to have the real numbers; it's impossible to do calculus in $\mathbb{Q}$ (the sequence 1, 1.4, 1.41, 1.414, is considered "not convergent"!)

Now fast-forward to 2002, and suppose you are given

Problem from USA TST 2002. Estimate the sum

$$f_p(x) = \sum_{k=1}^{p-1} \frac{1}{(px+k)^2}$$

to within mod p^3.

Even though $f_p(x)$ is a rational number, it still helps to be able to do analysis with infinite sums, and then bound the error term (i.e. take mod p^3). But the space $\mathbb{Q}$ is not complete with respect to $|\bullet|_p$ either, and thus it makes sense to work in the completion of $\mathbb{Q}$ with respect to $|\bullet|_p$. This is exactly $\mathbb{Q}_p$.

In any case, let's finally solve Example 27.1.1.

Example 27.3.9 (USA TST 2002)

We will now compute

$$f_p(x) = \sum_{k=1}^{p-1} \frac{1}{(px+k)^2} \pmod{p^3}.$$

Armed with the generalized binomial theorem, this becomes straightforward.

$$\begin{aligned} f_p(x) &= \sum_{k=1}^{p-1} \frac{1}{(px+k)^2} = \sum_{k=1}^{p-1} \frac{1}{k^2}\left(1+\frac{px}{k}\right)^{-2} \\ &= \sum_{k=1}^{p-1} \frac{1}{k^2} \sum_{n\ge 0} \binom{-2}{n} \left(\frac{px}{k}\right)^n \\ &= \sum_{n\ge 0} \binom{-2}{n} \sum_{k=1}^{p-1} \frac{1}{k^2} \left(\frac{x}{k}\right)^n p^n \\ &\equiv \sum_{k=1}^{p-1} \frac{1}{k^2} - 2x \left(\sum_{k=1}^{p-1} \frac{1}{k^3}\right) p + 3x^2 \left(\sum_{k=1}^{p-1} \frac{1}{k^4}\right) p^2 \pmod{p^3}. \end{aligned}$$

Using the elementary facts that $p^2 \mid \sum_k k^{-3}$ and $p \mid \sum_k k^{-4}$, this solves the problem.

§27.4 Mahler coefficients

One of the big surprises of p-adic analysis is that:

We can basically describe all continuous functions $\mathbb{Z}_p \to \mathbb{Q}_p$.

They are given by a basis of functions

$$\binom{x}{n} := \frac{x(x-1)\dots(x-(n-1))}{n!}$$

in the following way.

Theorem 27.4.1 (Mahler; see [Sc07, Theorem 51.1, Exercise 51.b])
Let $f\colon \mathbb{Z}_p \to \mathbb{Q}_p$ be continuous, and define

$$a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(k). \tag{27.1}$$

Then $\lim_n a_n = 0$ and

$$f(x) = \sum_{n\ge 0} a_n \binom{x}{n}.$$

Conversely, if a_n is any sequence converging to zero, then $f(x) = \sum_{n\ge 0} a_n \binom{x}{n}$ defines a continuous function satisfying (27.1).

The a_i are called the *Mahler coefficients* of f.

Exercise 27.4.2. Last post we proved that if $f\colon \mathbb{Z}_p \to \mathbb{Q}_p$ is continuous and $f(n) = (-1)^n$ for every $n \in \mathbb{Z}_{\ge 0}$ then $p = 2$. Re-prove this using Mahler's theorem, and this time show conversely that a unique such f exists when $p = 2$.

You'll note that these are the same finite differences that one uses on polynomials in high school math contests, which is why they are also called "Mahler differences".

$$\begin{aligned} a_0 &= f(0) \\ a_1 &= f(1) - f(0) \\ a_2 &= f(2) - 2f(1) + f(0) \\ a_3 &= f(3) - 3f(2) + 3f(1) - f(0). \end{aligned}$$

Thus one can think of $a_n \to 0$ as saying that the values of $f(0)$, $f(1)$, ... behave like a polynomial modulo p^e for every $e \ge 0$.

The notion "analytic" also has a Mahler interpretation. First, the definition.

Definition 27.4.3. We say that a function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ is **analytic** if it has a power series expansion

$$\sum_{n \ge 0} c_n x^n \quad c_n \in \mathbb{Q}_p \qquad \text{converging for } x \in \mathbb{Z}_p.$$

Theorem 27.4.4 ([Sc07, Theorem 54.4])

The function $f(x) = \sum_{n \ge 0} a_n \binom{x}{n}$ is analytic if and only if

$$\lim_{n \to \infty} \frac{a_n}{n!} = 0.$$

Analytic functions also satisfy the following niceness result:

Theorem 27.4.5 (Strassmann's theorem)

Let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be analytic. Then f has finitely many zeros.

To give an application of these results, we will prove the following result, which was interesting even before p-adics came along!

Theorem 27.4.6 (Skolem-Mahler-Lech)

Let $(x_i)_{i \ge 0}$ be an integral linear recurrence, meaning $(x_i)_{i \ge 0}$ is a sequence of integers

$$x_n = c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k} \qquad n = 1, 2, \dots$$

holds for some choice of integers $c_1, \dots, c_k$. Then the set of indices $\{i \mid x_i = 0\}$ is eventually periodic.

Proof. According to the theory of linear recurrences, there exists a matrix A such that we can write x_i as a dot product

$$x_i = \left\langle A^i u, v \right\rangle.$$

Let p be a prime not dividing $\det A$. Let T be an integer such that $A^T \equiv \mathrm{id} \pmod{p}$ (with id denoting the identity matrix).

Fix any $0 \le r < N$. We will prove that either all the terms

$$f(n) = x_{nT+r} \qquad n = 0, 1, \dots$$

are zero, or at most finitely many of them are. This will conclude the proof.

Let $A^T = \mathrm{id} + pB$ for some integer matrix B. We have

$$\begin{aligned} f(n) &= \left\langle A^{nT+r}u, v \right\rangle = \langle (\mathrm{id} + pB)^n A^r u, v \rangle \\ &= \sum_{k \ge 0} \binom{n}{k} \cdot p^n \langle B^n A^r u, v \rangle \\ &= \sum_{k \ge 0} a_n \binom{n}{k} \qquad \text{where } a_n = p^n \langle B^n A^r u, v \rangle \in p^n \mathbb{Z}. \end{aligned}$$

Thus we have written f in Mahler form. Initially, we define $f \colon \mathbb{Z}_{\ge 0} \to \mathbb{Z}$, but by Mahler's theorem (since $\lim_n a_n = 0$) it follows that f extends to a function $f \colon \mathbb{Z}_p \to \mathbb{Q}_p$. Also, we can check that $\lim_n \frac{a_n}{n!} = 0$ hence f is even analytic.

Thus by Strassman's theorem, f is either identically zero, or else it has finitely many zeros, as desired. □

§27.5 A few harder problems to think about

Problem 27A† ($\mathbb{Z}_p$ is compact)**.** Show that $\mathbb{Q}_p$ is not compact, but $\mathbb{Z}_p$ is. (For the latter, I recommend using sequential continuity.)

Problem 27B† (Totally disconnected)**.** Show that both $\mathbb{Z}_p$ and $\mathbb{Q}_p$ are *totally disconnected*: there are no connected sets other than the empty set and singleton sets.

Problem 27C (Mentioned in MathOverflow)**.** Let p be a prime. Find a sequence q_1, q_2, ... of rational numbers such that:

- the sequence q_n converges to 0 in the real sense;
- the sequence q_n converges to 2021 in the p-adic sense.

Problem 27D (USA TST 2011)**.** Let p be a prime. We say that a sequence of integers $\{z_n\}_{n=0}^\infty$ is a *p-pod* if for each $e \ge 0$, there is an $N \ge 0$ such that whenever $m \ge N$, p^e divides the sum

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} z_k.$$

Prove that if both sequences $\{x_n\}_{n=0}^\infty$ and $\{y_n\}_{n=0}^\infty$ are p-pods, then the sequence $\{x_n y_n\}_{n=0}^\infty$ is a p-pod.

28 Differentiation

§28.1 Definition

Prototypical example for this section: x^3 *has derivative* $3x^2$.

I suspect most of you have seen this before, but:

Definition 28.1.1. Let U be an open subset[1] of $\mathbb{R}$ and let $f\colon U \to \mathbb{R}$ be a function. Let $p \in U$. We say f is **differentiable** at p if the limit[2]

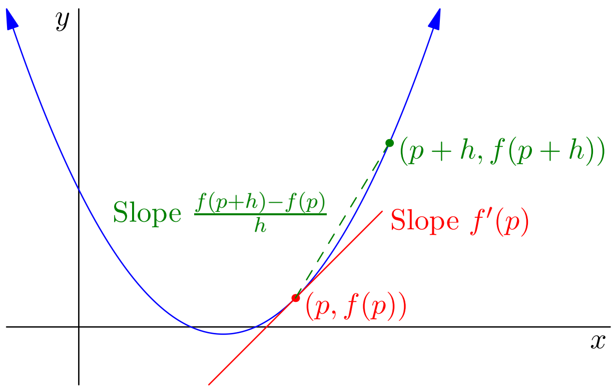
$$\lim_{h \to 0} \frac{f(p+h) - f(p)}{h}$$

exists. If so, we denote its value by $f'(p)$ and refer to this as the **derivative** of f at p.

The function f is differentiable if it is differentiable at every point. In that case, we regard the derivative $f'\colon (a,b) \to \mathbb{R}$ as a function it its own right.

> **Exercise 28.1.2.** Show that if f is differentiable at p then it is continuous at p too.

Here is the picture. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable (hence continuous). We draw a graph of f in the usual way and consider values of h. For any nonzero h, what we get is the slope of the *secant* line joining $(p, f(p))$ to $(p+h, f(p+h))$. However, as h gets close to zero, that secant line begins to approach a line which is tangent to the graph of the curve. A picture with f a parabola is shown below, with the tangent in red, and the secant in dashed green.



So the picture in your head should be that

$f'(p)$ looks like the slope of the tangent line at $(p, f(p))$.

[1] We will almost always use $U = (a,b)$ or $U = \mathbb{R}$, and you will not lose much by restricting the definition to those.

[2] Remember we are following the convention in Abuse of Notation 26.6.6. So we mean "the limit of the function $h \mapsto \frac{f(p+h)-f(p)}{h}$ except the value at $h = 0$ can be anything". And this is important because that fraction does not have a definition at $h = 0$. As promised, we pay this no attention.

Remark 28.1.3 — Note that the derivatives are defined for functions on *open* intervals. This is important. If $f\colon [a,b] \to \mathbb{R}$ for example, we could still define the derivative at each interior point, but $f'(a)$ no longer makes sense since f is not given a value on any open neighborhood of a.

Let's do one computation and get on with this.

Example 28.1.4 (Derivative of x^3 is $3x^2$)
Let $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$. For any point p, and *nonzero* h we can compute

$$\begin{aligned} \frac{f(p+h)-f(p)}{h} &= \frac{(p+h)^3 - p^3}{h} \\ &= \frac{3p^2h + 3ph^2 + h^3}{h} \\ &= 3p^2 + 3ph + h^2. \end{aligned}$$

Thus,

$$\lim_{h\to 0} \frac{f(p+h)-f(p)}{h} = \lim_{h\to 0}(3p^2 + 3ph + h^2) = 3p^2.$$

Thus the slope at each point of f is given by the formula $3p^2$. It is customary to then write $f'(x) = 3x^2$ as the derivative of the entire function f.

Abuse of Notation 28.1.5. We will now be sloppy and write this as $(x^3)' = 3x^2$. This is shorthand for the significantly more verbose "the real-valued function x^3 on domain so-and-so has derivative $3p^2$ at every point p in its domain".

In general, a real-valued differentiable function $f\colon U \to \mathbb{R}$ naturally gives rise to derivative $f'(p)$ at every point $p \in U$, so it is customary to just give up on p altogether and treat f' as function itself $U \to \mathbb{R}$, even though this real number is of a "different interpretation": $f'(p)$ is meant to interpret a slope (e.g. your hourly pay rate) as opposed to a value (e.g. your total dollar worth at time t). If f is a function from real life, the units do not even match!

This convention is so deeply entrenched I cannot uproot it without more confusion than it is worth. But if you read the chapters on multivariable calculus you will see how it comes back to bite us, when I need to re-define the derivative to be a *linear map*, rather than single real numbers.

§28.2 How to compute them

Same old, right? Sum rule, all that jazz.

Theorem 28.2.1 (Your friendly high school calculus rules)
In what follows f and g are differentiable functions, and U, V are open subsets of $\mathbb{R}$.

- (Sum rule) If $f, g\colon U \to \mathbb{R}$ then then $(f+g)'(x) = f'(x) + g'(x)$.
- (Product rule) If $f, g\colon U \to \mathbb{R}$ then then $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$.
- (Chain rule) If $f\colon U \to V$ and $g\colon V \to \mathbb{R}$ then the derivative of the composed function $g \circ f\colon U \to \mathbb{R}$ is $g'(f(x)) \cdot f'(x)$.

Proof.

- Sum rule: trivial, do it yourself if you care.
- Product rule: for every nonzero h and point $p \in U$ we may write
$$\frac{f(p+h)g(p+h)-f(p)g(p)}{h} = \frac{f(p+h)-f(p)}{h} \cdot g(p+h) + \frac{g(p+h)-g(p)}{h} \cdot f(p)$$
which as $h \to 0$ gives the desired expression.
- Chain rule: this is where Abuse of Notation 26.6.6 will actually bite us. Let $p \in U$, $q = f(p) \in V$, so that
$$(g \circ f)'(p) = \lim_{h \to 0} \frac{g(f(p+h))-g(q)}{h}.$$
We would like to write the expression in the limit as
$$\frac{g(f(p+h))-g(q)}{h} = \frac{g(f(p+h))-g(q)}{f(p+h)-q} \cdot \frac{f(p+h)-f(p)}{h}.$$
The problem is that the denominator $f(p+h)-f(p)$ might be zero. So instead, we define the expression
$$Q(y) = \begin{cases} \frac{g(y)-g(q)}{y-q} & \text{if } y \neq q \\ g'(q) & \text{if } y = q \end{cases}$$
which is continuous since g was differentiable at q. Then, we *do* have the equality
$$\frac{g(f(p+h))-g(q)}{h} = Q\left(f(p+h)\right) \cdot \frac{f(p+h)-f(p)}{h}.$$
because if $f(p+h) = q$ with $h \neq 0$, then both sides are equal to zero anyways.
Then, in the limit as $h \to 0$, we have $\lim_{h \to 0} \frac{f(p+h)-f(p)}{h} = f'(p)$, while $\lim_{h \to 0} Q(f(p+h)) = Q(q) = g'(q)$ by continuity. This was the desired result. □

Exercise 28.2.2. Compute the derivative of the polynomial $f(x) = x^3 + 10x^2 + 2019$, viewed as a function $f\colon \mathbb{R} \to \mathbb{R}$.

Remark 28.2.3 — Quick linguistic point: the theorems above all hold at each individual point. For example the sum rule really should say that if $f, g\colon U \to \mathbb{R}$ are differentiable at the point p then so is $f+g$ and the derivative equals $f'(p)+g'(p)$. Thus if f and g are differentiable on all of U, then it of course follows that $(f+g)' = f'+g'$. So each of the above rules has a "point-by-point" form which then implies the "whole U" form.

We only state the latter since that is what is used in practice. However, in the rare situations where you have a function differentiable only at certain points of U rather than the whole interval U, you can still use the below.

We next list some derivatives of well-known functions, but as we do not give rigorous definitions of these functions, we do not prove these here.

Proposition 28.2.4 (Derivatives of some well-known functions)

- The exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$ defined by $\exp(x) = e^x$ is its own derivative.
- The trig functions sin and cos have $\sin' = \cos$, $\cos' = -\sin$.

Example 28.2.5 (A typical high-school calculus question)

This means that you can mechanically compute the derivatives of any artificial function obtained by using the above, which makes it a great source of busy work in American high schools and universities. For example, if

$$f(x) = e^x + x\sin(x^2) \qquad f\colon \mathbb{R} \to \mathbb{R}$$

then one can compute f' by:

$$\begin{aligned} f'(x) &= (e^x)' + (x\sin(x^2))' && \text{sum rule} \\ &= e^x + (x\sin(x^2))' && \text{above table} \\ &= e^x + (x)'\sin(x^2) + x(\sin(x^2))' && \text{product rule} \\ &= e^x + \sin(x^2) + x(\sin(x^2))' && (x)' = 1 \\ &= e^x + \sin(x^2) + x \cdot 2x \cdot \cos(x^2) && \text{chain rule.} \end{aligned}$$

Of course, this function f is totally artificial and has no meaning, which is why calculus is the topic of widespread scorn in the United States. That said, it is worth appreciating that calculations like this are possible: one could say we have a pseudo-theorem "derivatives can actually be computed in practice".

If we take for granted that $(e^x)' = e^x$, then we can derive two more useful functions to add to our library of functions we can differentiate.

Corollary 28.2.6 (Power rule)

Let r be a real number. The function $\mathbb{R}_{>0} \to \mathbb{R}$ by $x \mapsto x^r$ has derivative $(x^r)' = rx^{r-1}$.

Proof. We knew this for integers r already, but now we can prove it for any positive real number r. Write

$$f(x) = x^r = e^{r \log x}$$

considered as a function $f\colon \mathbb{R}_{>0} \to \mathbb{R}$. The chain rule (together with the fact that $(e^x)' = e^x$) now gives

$$\begin{aligned} f'(x) &= e^{r \log x} \cdot (r \log x)' \\ &= e^{r \log x} \cdot \frac{r}{x} = x^r \cdot \frac{r}{x} = rx^{r-1}. \end{aligned}$$

The reason we don't prove the formulas for e^x and $\log x$ is that we don't at the moment even have a rigorous definition for either, or even for 2^x if x is not rational. However it's nice to know that some things imply the other. $\square$

Corollary 28.2.7 (Derivative of $\log$ is $1/x$)
The function $\log\colon \mathbb{R}_{>0} \to \mathbb{R}$ has derivative $(\log x)' = 1/x$.

Proof. We have that $x = e^{\log x}$. Differentiate both sides, and again use the chain rule[3]

$$1 = e^{\log x} \cdot (\log x)'.$$

Thus $(\log x)' = \frac{1}{e^{\log x}} = 1/x$. □

§28.3 Local (and global) maximums

Prototypical example for this section: Horizontal tangent lines to the parabola are typically good pictures.

You may remember from high school that one classical use of calculus was to extract the minimum or maximum values of functions. We will give a rigorous description of how to do this here.

Definition 28.3.1. Let $f\colon U \to \mathbb{R}$ be a function. A **local maximum** is a point $p \in U$ such that there exists an open neighborhood V of p (contained inside U) such that $f(p) \ge f(x)$ for every $x \in V$.

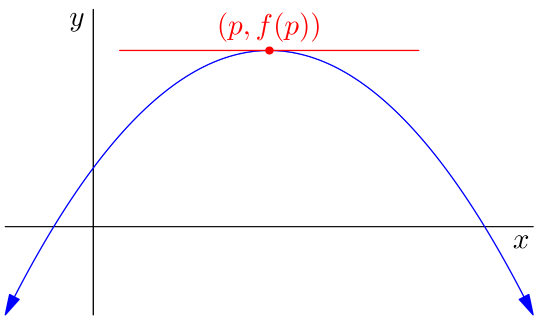
A **local minimum** is defined similarly.[4]

Definition 28.3.2. A point p is a **local extrema** if it satisfies either of these.

The nice thing about derivatives is that they pick up all extrema.

Theorem 28.3.3 (Fermat's theorem on stationary points)
Suppose $f\colon U \to \mathbb{R}$ is differentiable and $p \in U$ is a local extrema. Then $f'(p) = 0$.

If you draw a picture, this result is not surprising.



(Note also: the converse is not true. Say, $f(x) = x^{2019}$ has $f'(0) = 0$ but $x = 0$ is not a local extrema for f.)

Proof. Assume for contradiction $f'(p) > 0$. Choose any $\varepsilon > 0$ with $\varepsilon < f'(p)$. Then for sufficiently small $|h|$ we should have

$$\frac{f(p+h) - f(p)}{h} > \varepsilon.$$

[3]There is actually a small subtlety here: we are taking for granted that log is differentiable.
[4]Equivalently, it is a local maximum of $-f$.

In particular $f(p+h) > f(p)$ for $h > 0$ while $f(p-h) < f(p)$ for $h < 0$. So p is not a local extremum.

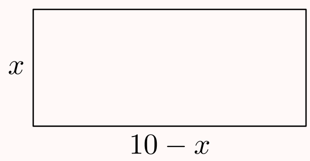
The proof for $f'(p) < 0$ is similar. $\square$

However, this is not actually adequate if we want a complete method for optimization. The issue is that we seek *global* extrema, which may not even exist: for example $f(x) = x$ (which has $f'(x) = 1$) obviously has no local extrema at all. The key to resolving this is to use *compactness*: we change the domain to be a compact set Z, for which we know that f will achieve some global maximum. The set Z will naturally have some *interior* S, and calculus will give us all the extrema within S. Then we manually check all cases outside Z.

Let's see two extended examples. The one is simple, and you probably already know about it, but I want to show you how to use compactness to argue thoroughly, and how the "boundary" points naturally show up.

Example 28.3.4 (Rectangle area optimization)

Suppose we consider rectangles with perimeter 20 and want the rectangle with the smallest or largest area.



If we choose the legs of the rectangle to be x and $10 - x$, then we are trying to optimize the function

$$f(x) = x(10-x) = 10x - x^2 \qquad f\colon [0, 10] \to \mathbb{R}.$$

By compactness, there exists *some* global maximum and *some* global minimum.

As f is differentiable on $(0, 10)$, we find that for any $p \in (0, 10)$, a global maximum will be a local maximum too, and hence should satisfy

$$0 = f'(p) = 10 - 2p \implies p = 5.$$

Also, the points $x = 0$ and $x = 10$ lie in the domain but not the interior $(0, 10)$. Therefore the global extrema (in addition to existing) must be among the three suspects $\{0, 5, 10\}$.

We finally check $f(0) = 0$, $f(5) = 25$, $f(10) = 0$. So the 5×5 square has the largest area and the degenerate rectangles have the smallest (zero) area.

Here is a non-elementary example.

Proposition 28.3.5 ($e^x \ge 1 + x$)

For all real numbers x we have $e^x \ge 1 + x$.

Proof. Define the differentiable function

$$f(x) = e^x - (x+1) \qquad f\colon \mathbb{R} \to \mathbb{R}.$$

Consider the compact interval $Z = [-1, 100]$. If $x \le -1$ then obviously $f(x) > 0$. Similarly if $x \ge 100$ then obviously $f(x) > 0$ too. So we just want to prove that if $x \in Z$, we have $f(x) \ge 0$.

Indeed, there exists *some* global minimum p. It could be the endpoints -1 or 100. Otherwise, if it lies in $U = (-1, 100)$ then it would have to satisfy

$$0 = f'(p) = e^p - 1 \implies p = 0.$$

As $f(-1) > 0$, $f(100) > 0$, $f(0) = 0$, we conclude $p = 0$ is the global minimum of Z; and hence $f(x) \ge 0$ for all $x \in Z$, hence for all x. □

Remark 28.3.6 — If you are willing to use limits at $\pm\infty$, you can rewrite proofs like the above in such a way that you don't have to explicitly come up with endpoints like -1 or 100. We won't do so here, but it's nice food for thought.

§28.4 Rolle and friends

Prototypical example for this section: The racetrack principle, perhaps?

One corollary of the work in the previous section is Rolle's theorem.

Theorem 28.4.1 (Rolle's theorem)

Suppose $f\colon [a, b] \to \mathbb{R}$ is a continuous function, which is differentiable on the open interval (a, b), such that $f(a) = f(b)$. Then there is a point $c \in (a, b)$ such that $f'(c) = 0$.

Proof. Assume f is nonconstant (otherwise any c works). By compactness, there exists both a global maximum and minimum. As $f(a) = f(b)$, either the global maximum or the global minimum must lie inside the open interval (a, b), and then Fermat's theorem on stationary points finishes. □

I was going to draw a picture until I realized xkcd #2042 has one already.

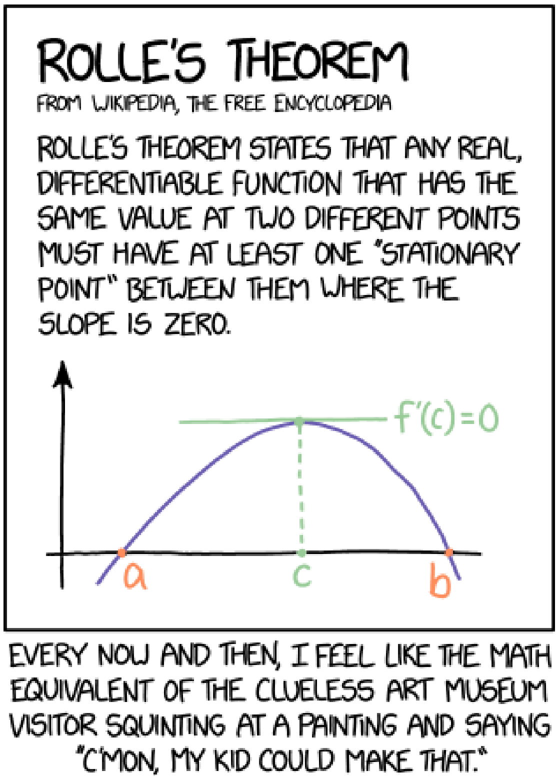


Image from [**Mu**]

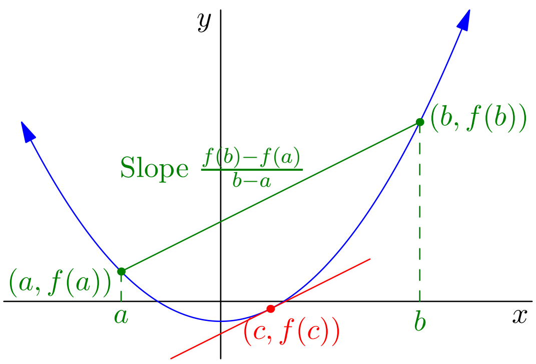
One can adapt the theorem as follows.

Theorem 28.4.2 (Mean value theorem)

Suppose $f \colon [a,b] \to \mathbb{R}$ is a continuous function, which is differentiable on the open interval (a,b). Then there is a point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Pictorially, there is a c such that the tangent at c has the same slope as the secant joining $(a, f(a))$, to $(b, f(b))$; and Rolle's theorem is the special case where that secant is horizontal.



Proof of mean value theorem. Let $s = \frac{f(b)-f(a)}{b-a}$ be the slope of the secant line, and define

$$g(x) = f(x) - sx$$

which intuitively shears f downwards so that the secant becomes vertical. In fact $g(a) = g(b)$ now, so we apply Rolle's theorem to g. □

Remark 28.4.3 (For people with driver's licenses) — There is a nice real-life interpretation of this I should mention. A car is travelling along a one-dimensional road (with $f(t)$ denoting the position at time t). Suppose you cover 900 kilometers in your car over the course of 5 hours (say $f(0) = 0$, $f(5) = 900$). Then there is *some* point at time in which your speed at that moment was exactly 180 kilometers per hour, and so you cannot really complain when the cops pull you over for speeding.

The mean value theorem is important because it lets you relate **use derivative information to get information about the function** in a way that is really not possible without it. Here is one quick application to illustrate my point:

Proposition 28.4.4 (Racetrack principle)
Let $f, g \colon \mathbb{R} \to \mathbb{R}$ be two differentiable functions with $f(0) = g(0)$.

(a) If $f'(x) \ge g'(x)$ for every $x > 0$, then $f(x) \ge g(x)$ for every $x > 0$.

(b) If $f'(x) > g'(x)$ for every $x > 0$, then $f(x) > g(x)$ for every $x > 0$.

This proposition might seem obvious. You can think of it as a race track for a reason: if f and g denote the positions of two cars (or horses etc) and the first car is always faster than the second car, then the first car should end up ahead of the second car. At a special case $g = 0$, this says that if $f'(x) \ge 0$, i.e. "f is increasing", then, well, $f(x) \ge f(0)$ for $x > 0$, which had better be true. However, if you try to prove this by definition from derivatives, you will find that it is not easy! However, it's almost a prototype for the mean value theorem.

Proof of racetrack principle. We prove (a). Let $h = f - g$, so $h(0) = 0$. Assume for contradiction $h(p) < 0$ for some $p > 0$. Then the secant joining $(0, h(0))$ to $(p, h(p))$ has negative slope; in other words by mean value theorem there is a $0 < c < p$ such that

$$f'(c) - g'(c) = h'(c) = \frac{h(p) - h(0)}{p} = \frac{h(p)}{p} < 0$$

so $f'(c) < g'(c)$, contradiction. Part (b) is the same. □

Sometimes you will be faced with two functions which you cannot easily decouple; the following form may be more useful in that case.

Theorem 28.4.5 (Ratio mean value theorem)
Let $f, g \colon [a, b] \to \mathbb{R}$ be two continuous functions which are differentiable on (a, b), and such that $g(a) \ne g(b)$. Then there exists $c \in (a, b)$ such that

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))$$

Proof. Use Rolle's theorem on the function

$$h(x) = [f(x) - f(a)]\,[g(b) - g(a)] - [g(x) - g(a)]\,[f(b) - f(a)]\,.$$ □

This is also called Cauchy's mean value theorem or the extended mean value theorem.

§28.5 Smooth functions

Prototypical example for this section: All the functions you're used to.

Let $f \colon U \to \mathbb{R}$ be differentiable, thus giving us a function $f' \colon U \to \mathbb{R}$. If our initial function was nice enough, then we can take the derivative again, giving a function $f'' \colon U \to \mathbb{R}$, and so on. In general, after taking the derivative n times, we denote the resulting function by $f^{(n)}$. By convention, $f^{(0)} = f$.

Definition 28.5.1. A function $f \colon U \to \mathbb{R}$ is **smooth** if it is infinitely differentiable; that is the function $f^{(n)}$ exists for all n.

> **Question 28.5.2.** Show that the absolute value function is not smooth.

Most of the functions we encounter, such as polynomials, e^x, log, sin, cos are smooth, and so are their compositions. Here is a weird example which we'll grow more next time.

Example 28.5.3 (A smooth function with all derivatives zero)

Consider the function

$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \le 0. \end{cases}$$

This function can be shown to be smooth, with $f^{(n)}(0) = 0$. So this function has every derivative at the origin equal to zero, despite being nonconstant!

§28.6 A few harder problems to think about

Problem 28A (Quotient rule)**.** Let $f \colon (a,b) \to \mathbb{R}$ and $g \colon (a,b) \to \mathbb{R}_{>0}$ be differentiable functions. Let $h = f/g$ be their quotient (also a function $(a,b) \to \mathbb{R}$). Show that the derivative of h is given by

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Problem 28B. For real numbers $x > 0$, how small can x^x be?

Problem 28C (RMM 2018)**.** Determine whether or not there exist nonconstant polynomials $P(x)$ and $Q(x)$ with real coefficients satisfying

$$P(x)^{10} + P(x)^9 = Q(x)^{21} + Q(x)^{20}.$$

Problem 28D. Let $P(x)$ be a degree n polynomial with real coefficients. Prove that the equation $e^x = P(x)$ has at most $n+1$ real solutions in x.

Problem 28E (Jensen's inequality)**.** Let $f \colon (a,b) \to \mathbb{R}$ be a twice differentiable function such that $f''(x) \ge 0$ for all x (i.e. f is *convex*). Prove that

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}$$

for all real numbers x and y in the interval (a,b).

Problem 28F (L'Hôpital rule, or at least one case)**.** Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be differentiable functions and let p be a real number. Suppose that

$$\lim_{x \to p} f(x) = \lim_{x \to p} g(x) = 0.$$

Prove that

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{f'(x)}{g'(x)}$$

provided the right-hand limit exists.

Problem 28G. Calculate the derivative of the function $f\colon (0, \infty) \to \mathbb{R}$ defined by $f(x) = x^x$.

29 Power series and Taylor series

Polynomials are very well-behaved functions, and are studied extensively for that reason. From an analytic perspective, for example, they are smooth, and their derivatives are easy to compute.

In this chapter we will study *power series*, which are literally "infinite polynomials" $\sum_n a_n x^n$. Armed with our understanding of series and differentiation, we will see three great things:

- Many of the functions we see in nature actually *are* given by power series. Among them are e^x, $\log x$, $\sin x$.
- Their convergence properties are actually quite well behaved: from the string of coefficients, we can figure out which x they converge for.
- The derivative of $\sum_n a_n x^n$ is actually just $\sum_n n a_n x^{n-1}$.

§29.1 Motivation

To get the ball rolling, let's start with one infinite polynomial you'll recognize: for any fixed number $-1 < x < 1$ we have the series convergence

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

by the geometric series formula.

Let's pretend we didn't see this already in Problem 26D. So, we instead have a smooth function $f\colon (-1,1) \to \mathbb{R}$ by

$$f(x) = \frac{1}{1-x}.$$

Suppose we wanted to pretend that it was equal to an "infinite polynomial" near the origin, that is

$$(1-x)^{-1} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots .$$

How could we find that polynomial, if we didn't already know?

Well, for starters we can first note that by plugging in $x = 0$ we obviously want $a_0 = 1$.

We have derivatives, so actually, we can then differentiate both sides to obtain that

$$(1-x)^{-2} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots .$$

If we now set $x = 0$, we get $a_1 = 1$. In fact, let's keep taking derivatives and see what we get.

$$\begin{aligned}
(1-x)^{-1} &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots \\
(1-x)^{-2} &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \dots \\
2(1-x)^{-3} &= 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \dots \\
6(1-x)^{-4} &= 6a_3 + 24a_4 x + 60a_5 x^2 + \dots \\
24(1-x)^{-5} &= 24a_4 + 120a_5 x + \dots \\
&\vdots .
\end{aligned}$$

If we set $x = 0$ we find $1 = a_0 = a_1 = a_2 = \dots$ which is what we expect; the geometric series $\frac{1}{1-x} = 1 + x + x^2 + \cdots$. And so actually taking derivatives was enough to get the right claim!

§29.2 Power series

Prototypical example for this section: $\frac{1}{1-z} = 1 + z + z^2 + \cdots$, *which converges on* $(-1, 1)$.

Of course this is not rigorous, since we haven't described what the right-hand side is, much less show that it can be differentiated term by term. So we define the main character now.

Definition 29.2.1. A **power series** is a sum of the form

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$$

where a_0, a_1, ... are real numbers, and z is a variable.

Abuse of Notation 29.2.2 ($0^0 = 1$)**.** If you are very careful, you might notice that when $z = 0$ and $n = 0$ we find 0^0 terms appearing. For this chapter the convention is that they are all equal to one.

Now, if I plug in a *particular* real number h, then I get a series of real numbers $\sum_{n=0}^{\infty} a_n h^n$. So I can ask, when does this series converge? It terms out there is a precise answer for this.

Definition 29.2.3. Given a power series $\sum_{n=0}^{\infty} a_n z^n$, the **radius of convergence** R is defined by the formula

$$\frac{1}{R} = \limsup_{n \to \infty} |a_n|^{1/n}.$$

with the convention that $R = 0$ if the right-hand side is ∞, and $R = \infty$ if the right-hand side is zero.

Theorem 29.2.4 (Cauchy-Hadamard theorem)

Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence R. Let h be a real number, and consider the infinite series

$$\sum_{n=0}^{\infty} a_n h^n$$

of real numbers. Then:

- The series converges absolutely if $|h| < R$.
- The series diverges if $|h| > R$.

Proof. This is not actually hard, but it won't be essential, so not included. □

Remark 29.2.5 — In the case $|h| = R$, it could go either way.

Example 29.2.6 ($\sum z^n$ has radius 1)
Consider the geometric series $\sum_n z^n = 1 + z + z^2 + \cdots$. Since $a_n = 1$ for every n, we get $R = 1$, which is what we expected.

Therefore, if $\sum_n a_n z^n$ is a power series with a nonzero radius $R > 0$ of convergence, then it can *also* be thought of as a function

$$(-R, R) \to \mathbb{R} \quad \text{by} \quad h \mapsto \sum_{n \ge 0} a_n h^n.$$

This is great. Note also that if $R = \infty$, this means we get a function $\mathbb{R} \to \mathbb{R}$.

Abuse of Notation 29.2.7 (Power series vs. functions)**.** There is some subtlety going on with "types" of objects again. Analogies with polynomials can help.

Consider $P(x) = x^3 + 7x + 9$, a polynomial. You *can*, for any real number h, plug in $P(h)$ to get a real number. However, in the polynomial *itself*, the symbol x is supposed to be a *variable* — which sometimes we will plug in a real number for, but that happens only after the polynomial is defined.

Despite this, "the polynomial $p(x) = x^3 + 7x + 9$" (which can be thought of as the coefficients) and "the real-valued function $x \mapsto x^3 + 7x + 9$" are often used interchangeably. The same is about to happen with power series: while they were initially thought of as a sequence of coefficients, the Cauchy-Hadamard theorem lets us think of them as functions too, and thus we blur the distinction between them.

§29.3 Differentiating them

Prototypical example for this section: We saw earlier $1 + x + x^2 + x^3 + \dots$ *has derivative* $1 + 2x + 3x^2 + \dots$.

As promised, differentiation works exactly as you want.

Theorem 29.3.1 (Differentiation works term by term)
Let $\sum_{n \ge 0} a_n z^n$ be a power series with radius of convergence $R > 0$, and consider the corresponding function

$$f \colon (-R, R) \to \mathbb{R} \quad \text{by} \quad f(x) = \sum_{n \ge 0} a_n x^n.$$

Then all the derivatives of f exist and are given by power series

$$\begin{aligned} f'(x) &= \sum_{n \ge 1} n a_n x^{n-1} \\ f''(x) &= \sum_{n \ge 2} n(n-1) a_n x^{n-2} \\ &\vdots \end{aligned}$$

which also converge for any $x \in (-R, R)$. In particular, f is smooth.

Proof. Also omitted. The right way to prove it is to define the notion "converges uniformly", and strengthen Cauchy-Hadamard to have this as a conclusion as well. □

Corollary 29.3.2 (A description of power series coefficients)

Let $\sum_{n\ge 0} a_n z^n$ be a power series with radius of convergence $R > 0$, and consider the corresponding function $f(x)$ as above. Then

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

Proof. Take the nth derivative and plug in $x = 0$. □

§29.4 Analytic functions

Prototypical example for this section: The piecewise $e^{-1/x}$ or 0 function is not analytic, but is smooth.

With all these nice results about power series, we now have a way to do this process the other way: suppose that $f : U \to \mathbb{R}$ is a function. Can we express it as a power series?

Functions for which this *is* true are called analytic.

Definition 29.4.1. A function $f : U \to \mathbb{R}$ is **analytic** at the point $p \in U$ if there exists an open neighborhood V of p (inside U) and a power series $\sum_n a_n z^n$ such that

$$f(x) = \sum_{n\ge 0} a_n (x-p)^n$$

for any $x \in V$. As usual, the whole function is analytic if it is analytic at each point.

Question 29.4.2. Show that if f is analytic, then it's smooth.

Moreover, if f is analytic, then by the corollary above its coefficients are actually described exactly by

$$f(x) = \sum_{n\ge 0} \frac{f^{(n)}(p)}{n!}(x-p)^n.$$

Even if f is smooth but not analytic, we can at least write down the power series; we give this a name.

Definition 29.4.3. For smooth f, the power series $\sum_{n\ge 0} \frac{f^{(n)}(p)}{n!} z^n$ is called the **Taylor series** of f at p.

Example 29.4.4 (Examples of analytic functions)

(a) Polynomials, sin, cos, e^x, log all turn out to be analytic.

(b) The smooth function from before defined by

$$f(x) = \begin{cases} \exp(-1/x) & x > 0 \\ 0 & x \le 0 \end{cases}$$

is *not* analytic. Indeed, suppose for contradiction it was. As all the derivatives are zero, its Taylor series would be $0 + 0x + 0x^2 + \cdots$. This Taylor series does *converge*, but not to the right value — as $f(\varepsilon) > 0$ for any $\varepsilon > 0$, contradiction.

Example (b) shows that if you have a function $f \colon \mathbb{R} \to \mathbb{R}$, then even knowing f is smooth and the full Taylor series at p, it's still impossible to recover any other values of f or deduce that f is analytic in any interval containing p.

However, it's at least true that:

Proposition 29.4.5 (Analytic at one point implies analytic on an interval)

Let $f \colon \mathbb{R} \to \mathbb{R}$ be smooth, and let $p \in \mathbb{R}$ be a point in the domain. Suppose that

- the Taylor series of f at p has radius of convergence $R > 0$; *and*
- that Taylor series actually does converge to the value $f(x)$ for every input $x \in (p-R, p+R)$ within the radius of convergence.

Then f is analytic on $(p-R, p+R)$.

This result is nontrivial because *a priori* we only know f is analytic at p; the result extends that to being analytic on the radius of convergence if $R > 0$. We'll use it for exp in just a moment, which is actually defined by a power series.

Like with differentiable functions:

Proposition 29.4.6 (All your usual closure properties for analytic functions)

The sums, products, compositions, nonzero quotients of analytic functions are analytic.

The upshot of this is that most of your usual functions that occur in nature, or even artificial ones like $f(x) = e^x + x\sin(x^2)$, will be analytic, hence describable locally by Taylor series.

§29.5 A definition of Euler's constant and exponentiation

We can actually give a definition of e^x using the tools we have now.

Definition 29.5.1. We define the map $\exp \colon \mathbb{R} \to \mathbb{R}$ by using the following power series, which has infinite radius of convergence:

$$\exp(x) = \sum_{n \geq 0} \frac{x^n}{n!}.$$

We then define Euler's constant as $e = \exp(1)$.

> **Question 29.5.2.** Show that under this definition, $\exp' = \exp$. Also conclude from Proposition 29.4.5 that exp is analytic.

We are then settled with:

Proposition 29.5.3 (exp is multiplicative)

Under this definition,

$$\exp(x+y) = \exp(x)\exp(y).$$

Idea of proof. There is some subtlety here with switching the order of summation that we won't address. Modulo that:

$$\begin{aligned} \exp(x)\exp(y) &= \sum_{n\ge 0}\frac{x^n}{n!}\sum_{m\ge 0}\frac{y^m}{m!} = \sum_{n\ge 0}\sum_{m\ge 0}\frac{x^n}{n!}\frac{y^m}{m!} \\ &= \sum_{k\ge 0}\sum_{\substack{m+n=k \\ m,n\ge 0}}\frac{x^n y^m}{n!m!} = \sum_{k\ge 0}\sum_{\substack{m+n=k \\ m,n\ge 0}}\binom{k}{n}\frac{x^n y^m}{k!} \\ &= \sum_{k\ge 0}\frac{(x+y)^k}{k!} = \exp(x+y). \end{aligned}$$ □

Corollary 29.5.4 (exp is positive)

(a) We have $\exp(x) > 0$ for any real number x.

(b) The function exp is strictly increasing.

Proof. First
$$\exp(x) = \exp(x/2)^2 \ge 0$$
which shows exp is nonnegative. Also, $1 = \exp(0) = \exp(x)\exp(-x)$ implies $\exp(x) \ne 0$ for any x, proving (a).

(b) is just since $\exp'$ is strictly positive (racetrack principle). □

The log function then comes after.

Definition 29.5.5. We may define $\log\colon \mathbb{R}_{>0} \to \mathbb{R}$ to be the inverse function of exp.

Since its derivative is $1/x$ it is smooth; and then one may compute its coefficients to show it is analytic.

Note that this actually gives us a rigorous way to define a^r for any $a > 0$ and $r > 0$, namely
$$a^r := \exp\left(r \log a\right).$$

§29.6 This all works over complex numbers as well, except also complex analysis is heaven

We now mention that every theorem we referred to above holds equally well if we work over $\mathbb{C}$, with essentially no modifications.

- Power series are defined by $\sum_n a_n z^n$ with $a_n \in \mathbb{C}$, rather than $a_n \in \mathbb{R}$.
- The definition of radius of convergence R is unchanged! The series will converge if $|z| < R$.
- Differentiation still works great. (The definition of the derivative is unchanged.)
- Analytic still works great for functions $f\colon U \to \mathbb{C}$, with $U \subseteq \mathbb{C}$ open.

In particular, we can now even define complex exponentials, giving us a function

$$\exp\colon \mathbb{C} \to \mathbb{C}$$

since the power series still has $R = \infty$. More generally if $a > 0$ and $z \in \mathbb{C}$ we may still define

$$a^z := \exp(z \log a).$$

(We still require the base a to be a positive real so that $\log a$ is defined, though. So this i^i issue is still there.)

However, if one tries to study calculus for complex functions as we did for the real case, in addition to most results carrying over, we run into a huge surprise:

If $f\colon \mathbb{C} \to \mathbb{C}$ is differentiable, it is analytic.

And this is just the beginning of the nearly unbelievable results that hold for complex analytic functions. But this is the part on real analysis, so you will have to read about this later!

§29.7 A few harder problems to think about

Problem 29A. Find the Taylor series of $\log(1-x)$.

Problem 29B† **(Euler formula).** Show that

$$\exp(i\theta) = \cos\theta + i\sin\theta$$

for any real number θ.

Problem 29C† **(Taylor's theorem, Lagrange form).** Let $f\colon [a,b] \to \mathbb{R}$ be continuous and $n+1$ times differentiable on (a,b). Define

$$P_n = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} \cdot (b-a)^k.$$

Prove that there exists $\xi \in (a,b)$ such that

$$f^{(n+1)}(\xi) = (n+1)! \cdot \frac{f(b) - P_n}{(b-a)^{n+1}}.$$

This generalizes the mean value theorem (which is the special case $n = 0$, where $P_0 = f(a)$).

Problem 29D (Putnam 2018 A5). Let $f\colon \mathbb{R} \to \mathbb{R}$ be smooth, and assume that $f(0) = 0$, $f(1) = 1$, and $f(x) \ge 0$ for every real number x. Prove that $f^{(n)}(x) < 0$ for some positive integer n and real number x.

Problem 29E. Let $f\colon \mathbb{R} \to \mathbb{R}$ be smooth. Suppose that for every point p, the Taylor series of f at p has positive radius of convergence. Prove that there exists at least one point at which f is analytic.

30 Riemann integrals

> "Trying to Riemann integrate discontinuous functions is kind of outdated."
> — Dennis Gaitsgory, [Ga15]

We will go ahead and define the Riemann integral, but we won't do very much with it. The reason is that the Lebesgue integral is basically better, so we will define it, check the fundamental theorem of calculus (or rather, leave it as a problem at the end of the chapter), and then always use Lebesgue integrals forever after.

§30.1 Uniform continuity

Prototypical example for this section: $f(x) = x^2$ *is not uniformly continuous on* $\mathbb{R}$*, but functions on compact sets are always uniformly continuous.*

Definition 30.1.1. Let $f\colon M \to N$ be a continuous map between two metric spaces. We say that f is **uniformly continuous** if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_M(p,q) < \delta \implies d_N(f(p), f(q)) < \varepsilon.$$

This difference is that given an $\varepsilon > 0$ we must specify a $\delta > 0$ which works for *every* choice p and q of inputs; whereas usually δ is allowed to depend on p and q. (Also, this definition can't be ported to a general topological space.)

Example 30.1.2 (Uniform continuity failure)

(a) The function $f\colon \mathbb{R} \to \mathbb{R}$ by $x \mapsto x^2$ is not uniformly continuous. Suppose we take $\varepsilon = 0.1$ for example. There is no δ such that if $|x-y| < \delta$ then $|x^2 - y^2| < 0.1$, since as x and y get large, the function f becomes increasingly sensitive to small changes.

(b) The function $(0,1) \to \mathbb{R}$ by $x \mapsto x^{-1}$ is not uniformly continuous.

(c) The function $\mathbb{R}_{>0} \to \mathbb{R}$ by $x \mapsto \sqrt{x}$ does turn out to be uniformly continuous (despite having unbounded derivatives!). Indeed, you can check that the assertion

$$|x-y| < \varepsilon^2 \implies \left|\sqrt{x} - \sqrt{y}\right| < \varepsilon$$

holds for any $x, y, \varepsilon > 0$.

The good news is that in the compact case all is well.

Theorem 30.1.3 (Uniform continuity free for compact spaces)

Let M be a compact metric space. Then any continuous map $f\colon M \to N$ is also uniformly continuous.

Proof. Assume for contradiction there is some bad $\varepsilon > 0$. Then taking $\delta = 1/n$, we find that for each integer n there exists points p_n and q_n which are within $1/n$ of each other, but are mapped more than ε away from each other by f. In symbols, $d_M(p_n, q_n) < 1/n$ but $d_N(f(p_n), f(q_n)) > \varepsilon$.

By compactness of M, we can find a convergent subsequence $p_{i_1}, p_{i_2}, \dots$ converging to some $x \in M$. Since the q_{i_n} is within $1/i_n$ of p_{i_n}, it ought to converge as well, to the same point $x \in M$. Then the sequences $f(p_{i_n})$ and $f(q_{i_n})$ should both converge to $f(x) \in N$, but this is impossible as they are always ε away from each other. $\square$

This means for example that x^2 viewed as a continuous function $[0,1] \to \mathbb{R}$ is automatically uniformly continuous. Man, isn't compactness great?

§30.2 Dense sets and extension

Prototypical example for this section: Functions from $\mathbb{Q} \to N$ extend to $\mathbb{R} \to N$ if they're uniformly continuous and N is complete. See also counterexamples below.

Definition 30.2.1. Let S be a subset (or subspace) of a topological space X. Then we say that S is **dense** if every open subset of X contains a point of S.

Example 30.2.2 (Dense sets)

(a) $\mathbb{Q}$ is dense in $\mathbb{R}$.

(b) In general, any metric space M is dense in its completion $\overline{M}$.

Dense sets lend themselves to having functions completed. The idea is that if I have a continuous function $f \colon \mathbb{Q} \to N$, for some metric space N, then there should be *at most* one way to extend it to a function $\widetilde{f} \colon \mathbb{R} \to N$. For we can approximate each rational number by real numbers: if I know $f(1)$, $f(1.4)$, $f(1.41)$, ... $\widetilde{f}(\sqrt{2})$ had better be the limit of this sequence. So it is certainly unique.

However, there are two ways this could go wrong:

Example 30.2.3 (Non-existence of extension)

(a) It could be that N is not complete, so the limit may not even exist in N. For example if $N = \mathbb{Q}$, then certainly there is no way to extend even the identity function $f \colon \mathbb{Q} \to N$ to a function $\widetilde{f} \colon \mathbb{R} \to N$.

(b) Even if N was complete, we might run into issues where f explodes. For example, let $N = \mathbb{R}$ and define

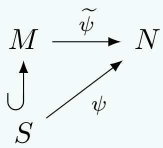
$$f(x) = \frac{1}{x - \sqrt{2}} \qquad f \colon \mathbb{Q} \to \mathbb{R}.$$

There is also no way to extend this due to the explosion of f near $\sqrt{2} \notin \mathbb{Q}$, which would cause $\widetilde{f}(\sqrt{2})$ to be undefined.

However, the way to fix this is to require f to be uniformly continuous, and in that case we do get a unique extension.

Theorem 30.2.4 (Extending uniformly continuous functions)

Let M be a metric space, N a *complete* metric space, and S a dense subspace of M. Suppose $\psi: S \to N$ is a *uniformly* continuous function. Then there exists a unique continuous function $\widetilde{\psi}: M \to N$ such that the diagram



commutes.

Outline of proof. As mentioned in the discussion, each $x \in M$ can be approximated by a sequence $x_1, x_2, \dots$ in S with $x_i \to x$. The two main hypotheses, completeness and uniform continuity, are now used:

> **Exercise 30.2.5.** Prove that $\psi(x_1), \psi(x_2), \dots$ converges in N by using uniform continuity to show that it is Cauchy, and then appealing to completeness of N.

Hence we define $\widetilde{\psi}(x)$ to be the limit of that sequence; this doesn't depend on the choice of sequence, and one can use sequential continuity to show $\widetilde{\psi}$ is continuous. □

§30.3 Defining the Riemann integral

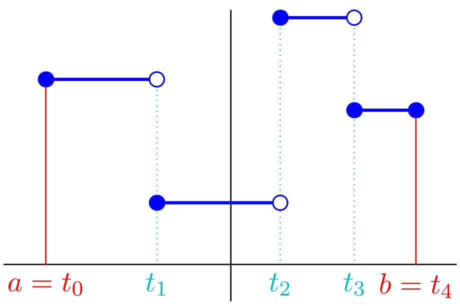
Extensions will allow us to define the Riemann integral. I need to introduce a bit of notation so bear with me.

Definition 30.3.1. Let $[a, b]$ be a closed interval.

- We let $C^0([a, b])$ denote the set of continuous functions on $[a, b] \to \mathbb{R}$.
- We let $R([a, b])$ denote the set of **rectangle functions** on $[a, b] \to \mathbb{R}$. These functions which are constant on the intervals $[t_0, t_1)$, $[t_1, t_2)$, $[t_2, t_3)$, $\dots$, $[t_{n-2}, t_{n-1})$, and also $[t_{n-1}, t_n]$, for some $a = t_0 < t_1 < t_2 < \dots < t_n = b$.
- We let $M([a, b]) = C^0([a, b]) \cup R([a, b])$.

Warning: only $C^0([a, b])$ is common notation, and the other two are made up.

See picture below for a typical a rectangle function. (It is irritating that we have to officially assign a single value to each t_i, even though there are naturally two values we want to use, and so we use the convention of letting the left endpoint be closed).



Definition 30.3.2. We can impose a metric on $M([a,b])$ by defining

$$d(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|.$$

Now, there is a natural notion of integral for rectangle functions: just sum up the obvious rectangles! Officially, this is the expression

$$f(a)(t_1 - a) + f(t_1)(t_2 - t_1) + +f(t_2)\,(t_3 - t_2) + \cdots + f(t_n)\,(b - t_n)\,.$$

We denote this function by

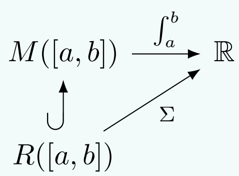
$$\Sigma \colon R([a,b]) \to \mathbb{R}.$$

Theorem 30.3.3 (The Riemann integral)

There exists a unique continuous map

$$\textstyle\int_a^b \colon M([a,b]) \to \mathbb{R}$$

such that the diagram

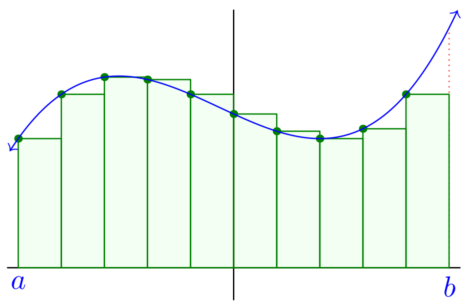


commutes.

Proof. We want to apply the extension theorem, so we just have to check a few things:

- We claim $R([a,b])$ is a dense subset of $M([a,b])$. In other words, for any continuous $f \colon [a,b] \to \mathbb{R}$ and $\varepsilon > 0$, we want there to exist a rectangle function that approximates f within ε.

  This follows by uniform continuity. We know there exists a $\delta > 0$ such that whenever $|x-y| < \delta$ we have $|f(x) - f(y)| < \varepsilon$. So as long as we select a rectangle function whose rectangles have width less than δ, and such that the upper-left corner of each rectangle lies on the graph of f, then we are all set.



- The "add-the-rectangles" map $\Sigma \colon R([a,b]) \to \mathbb{R}$ is *uniformly* continuous. Actually this is pretty obvious: if two rectangle functions f and g have $d(f,g) < \varepsilon$, then $d(\Sigma f, \Sigma g) < \varepsilon(b-a)$.

- $\mathbb{R}$ is complete. $\square$

§30.4 Meshes

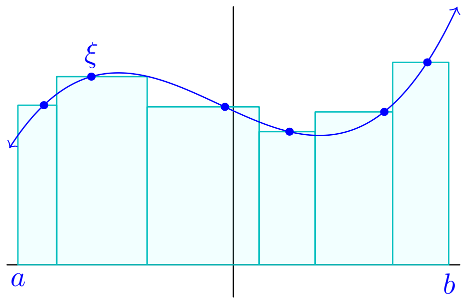
The above definition might seem fantastical, overcomplicated, hilarious, or terrible, depending on your taste. But if you unravel it, it's really the picture you are used to. What we have done is taking every continuous function $f\colon [a,b] \to \mathbb{R}$ and showed that it can be approximated by a rectangle function (which we phrased as a dense inclusion). Then we added the area of the rectangles. Nonetheless, we will give a definition that's more like what you're used to seeing in other places.

Definition 30.4.1. A *tagged partition* P of $[a,b]$ consists of a partition of $[a,b]$ into n intervals, with a point ξ_i in the nth interval, denoted

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b \qquad \text{and} \qquad \xi_i \in [t_{i-1}, t_i] \quad \forall\, 1 \le i \le n.$$

The *mesh* of P is the width of the longest interval, i.e. $\max_i(t_i - t_{i-1})$.

Of course the point of this definition is that we add the rectangles, but the ξ_i are the sample points.



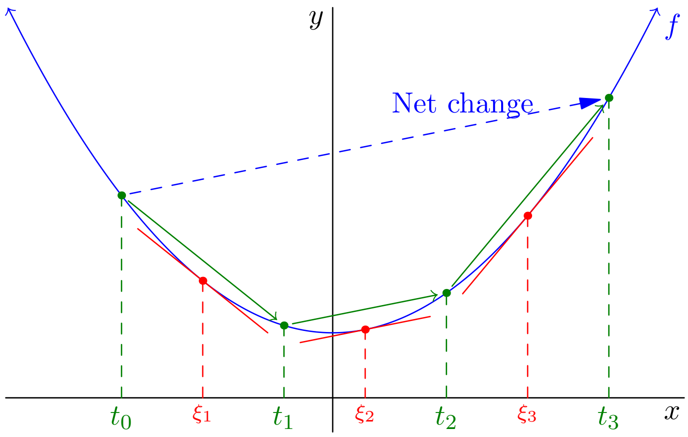
Theorem 30.4.2 (Riemann integral)
Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then

$$\int_a^b f(x)\,dx = \lim_{\substack{P \text{ tagged partition} \\ \text{mesh } P \to 0}} \left(\sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) \right).$$

Here the limit means that we can take any sequence of partitions whose mesh approaches zero.

Proof. The right-hand side corresponds to the areas of some rectangle functions g_1, g_2, ... with increasingly narrow rectangles. As in the proof Theorem 30.3.3, as the meshes of those rectangles approaches zero, by uniform continuity, we have $d(f, g_n) \to 0$ as well. Thus by continuity in the diagram of Theorem 30.3.3, we get $\lim_n \Sigma(g_n) = \int(f)$ as needed. □

Combined with the mean value theorem, this can be used to give a short proof of the fundamental theorem of calculus for functions f with a continuous derivative. The idea is that for any choice of partition $a \le t_0 < t_1 < t_2 < \cdots < t_n \le b$, using the Mean Value Theorem it should be possible to pick ξ_i in each interval to match with the slope of the secant: at which point the areas sum to the total change in f. We illustrate this situation with three points, and invite the reader to fill in the details as Problem 30B*.



One quick note is that although I've only defined the Riemann integral for continuous functions, there ought to be other functions for which it exists (including "piecewise continuous functions" for example, or functions "continuous almost everywhere"). The relevant definition is:

Definition 30.4.3. If $f\colon [a,b] \to \mathbb{R}$ is a function which is not necessarily continuous, but for which the limit

$$\lim_{\substack{P \text{ tagged partition} \\ \text{mesh } P \to 0}} \left(\sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}) \right).$$

exists anyways, then we say f is **Riemann integrable** on $[a,b]$ and define its value to be that limit $\int_a^b f(x)\,dx$.

We won't really use this definition much, because we will see that every Riemann integrable function is Lebesgue integrable, and the Lebesgue integral is better.

Example 30.4.4 (Your AP calculus returns)

We had better mention that Problem 30B* implies that we can compute Riemann integrals in practice, although most of you may already know this from high-school calculus. For example, on the interval $(1,4)$, the derivative of the function $F(x) = \frac{1}{3}x^3$ is $F'(x) = x^2$. As $f(x) = x^2$ is a continuous function $f\colon [1,4] \to \mathbb{R}$, we get

$$\int_1^4 x^2\,dx = F(4) - F(1) = \frac{64}{3} - \frac{1}{3} = 21.$$

Note that we could also have picked $F(x) = \frac{1}{3}x^3 + 2019$; the function F is unique up to shifting, and this constant cancels out when we subtract. This is why it's common in high school to (really) abuse notation and write $\int x^2\,dx = \frac{1}{3}x^3 + C$.

§30.5 A few harder problems to think about

Problem 30A. Let $f\colon (a,b) \to \mathbb{R}$ be differentiable and assume f' is bounded. Show that f is uniformly continuous.

Problem 30B* (Fundamental theorem of calculus)**.** Let $f\colon [a,b] \to \mathbb{R}$ be continuous, differentiable on (a,b), and assume the derivative f' extends to a continuous function $f'\colon [a,b] \to \mathbb{R}$. Prove that

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Problem 30C* (Improper integrals)**.** For each real number $r > 0$, evaluate the limit[1]

$$\lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \frac{1}{x^r}\, dx$$

or show it does not exist.

This can intuitively be thought of as the "improper" integral $\int_0^1 x^{-r}\, dx$; it doesn't make sense in our original definition since we did not (and cannot) define the integral over the non-compact $(0, 1]$ but we can still consider the integral over $[\varepsilon, 1]$ for any $\varepsilon > 0$.

Problem 30D. Show that

$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right) = \log 2.$$

[1] If you are not familiar with the notation $\varepsilon \to 0^+$, you can replace ε with $1/M$ for $M > 0$, and let $M \to \infty$ instead.