III

Basic Topology

Part III: Contents

6 Properties of metric spaces

At the end of the last chapter on metric spaces, we introduced two adjectives "open" and "closed". These are important because they'll grow up to be the *definition* for a general topological space, once we graduate from metric spaces.

To move forward, we provide a couple niceness adjectives that applies to *entire metric spaces*, rather than just a set relative to a parent space. They are "(totally) bounded" and "complete". These adjectives are specific to metric spaces, but will grow up to become the notion of *compactness*, which is, in the words of [Pu02], "the single most important concept in real analysis". At the end of the chapter, we will know enough to realize that something is amiss with our definition of homeomorphism, and this will serve as the starting point for the next chapter, when we define fully general topological spaces.

§6.1 Boundedness

Prototypical example for this section: $[0,1]$ *is bounded but* $\mathbb{R}$ *is not.*

Here is one notion of how to prevent a metric space from being a bit too large.

Definition 6.1.1. A metric space M is **bounded** if there is a constant D such that $d(p,q) \le D$ for all $p, q \in M$.

You can change the order of the quantifiers:

Proposition 6.1.2 (Boundedness with radii instead of diameters)
A metric space M is bounded if and only if for every point $p \in M$, there is a radius R (possibly depending on p) such that $d(p,q) \le R$ for all $q \in M$.

Exercise 6.1.3. Use the triangle inequality to show these are equivalent. (The names "radius" and "diameter" are a big hint!)

Example 6.1.4 (Examples of bounded spaces)

(a) Finite intervals like $[0,1]$ and (a,b) are bounded.

(b) The unit square $[0,1]^2$ is bounded.

(c) $\mathbb{R}^n$ is not bounded for any $n \ge 1$.

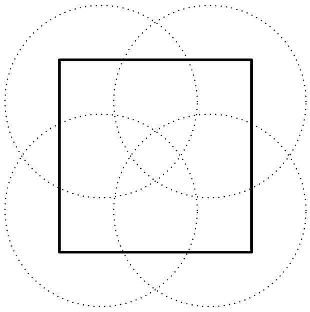
(d) A discrete space on an infinite set is bounded.

(e) $\mathbb{N}$ is not bounded, despite being homeomorphic to the discrete space!

The fact that a discrete space on an infinite set is "bounded" might be upsetting to you, so here is a somewhat stronger condition you can use:

Definition 6.1.5. A metric space is **totally bounded** if for any $\varepsilon > 0$, we can cover M with finitely many ε-neighborhoods.

For example, if $\varepsilon = 1/2$, you can cover $[0,1]^2$ by ε-neighborhoods.



Exercise 6.1.6. Show that "totally bounded" implies "bounded".

Example 6.1.7 (Examples of totally bounded spaces)

(a) A subset of $\mathbb{R}^n$ is bounded if and only if it is totally bounded.

This is for Euclidean geometry reasons: for example in $\mathbb{R}^2$ if I can cover a set by a single disk of radius 2, then I can certainly cover it by finitely many disks of radius 1/2. (We won't prove this rigorously.)

(b) So for example $[0,1]$ or $[0,2] \times [0,3]$ is totally bounded.

(c) In contrast, a discrete space on an infinite set is not totally bounded.

§6.2 Completeness

Prototypical example for this section: $\mathbb{R}$ *is complete, but* $\mathbb{Q}$ *and* $(0,1)$ *are not.*

So far we can only talk about sequences converging if they have a limit. But consider the sequence

$$x_1 = 1, \; x_2 = 1.4, \; x_3 = 1.41, \; x_4 = 1.414, \dots .$$

It converges to $\sqrt{2}$ in $\mathbb{R}$, of course. But it fails to converge in $\mathbb{Q}$; there is no *rational* number this sequence converges to. And so somehow, if we didn't know about the existence of $\mathbb{R}$, we would have *no idea* that the sequence (x_n) is "approaching" something.

That seems to be a shame. Let's set up a new definition to describe these sequences whose terms **get close to each other**, even if they don't approach any particular point in the space. Thus, we only want to mention the given points in the definition.

Definition 6.2.1. Let $x_1, x_2, \dots$ be a sequence which lives in a metric space $M = (M, d_M)$. We say the sequence is **Cauchy** if for any $\varepsilon > 0$, we have

$$d_M(x_m, x_n) < \varepsilon$$

for all sufficiently large m and n.

Question 6.2.2. Show that a sequence which converges is automatically Cauchy. (Draw a picture.)

Now we can define:

Definition 6.2.3. A metric space M is **complete** if every Cauchy sequence converges.

Example 6.2.4 (Examples of complete spaces)

(a) $\mathbb{R}$ is complete. (Depending on your definition of $\mathbb{R}$, this either follows by definition, or requires some work. We won't go through this here.)

(b) The discrete space is complete, as the only Cauchy sequences are eventually constant.

(c) The closed interval $[0, 1]$ is complete.

(d) $\mathbb{R}^n$ is complete as well. (You're welcome to prove this by induction on n.)

Example 6.2.5 (Non-examples of complete spaces)

(a) The rationals $\mathbb{Q}$ are not complete.

(b) The open interval $(0, 1)$ is not complete, as the sequence 0.9, 0.99, 0.999, 0.9999, ... is Cauchy but does not converge.

So, metric spaces need not be complete, like $\mathbb{Q}$. But we certainly would like them to be complete, and in light of the following theorem this is not unreasonable.

Theorem 6.2.6 (Completion)

Every metric space can be "completed", i.e. made into a complete space by adding in some points.

We won't need this construction at all, so it's left as Problem 6C†.

Example 6.2.7 ($\mathbb{Q}$ completes to $\mathbb{R}$)

The completion of $\mathbb{Q}$ is $\mathbb{R}$.

(In fact, by using a modified definition of completion not depending on the real numbers, other authors often use this as the definition of $\mathbb{R}$.)

§6.3 Let the buyer beware

There is something suspicious about both these notions: neither are preserved under homeomorphism!

Example 6.3.1 (Something fishy is going on here)

Let $M = (0, 1)$ and $N = \mathbb{R}$. As we saw much earlier M and N are homeomorphic. However:

- $(0, 1)$ is totally bounded, but not complete.
- $\mathbb{R}$ is complete, but not bounded.

This is the first hint of something going awry with the metric. As we progress further into our study of topology, we will see that in fact *open sets and closed sets* (which we motivated by using the metric) are the notion that will really shine later on. I insist on introducing the metric first so that the standard pictures of open sets and closed sets make sense, but eventually it becomes time to remove the training wheels.

§6.4 Subspaces, and (inb4) a confusing linguistic point

Prototypical example for this section: A circle is obtained as a subspace of $\mathbb{R}^2$.

As we've already been doing implicitly in examples, we'll now say:

Definition 6.4.1. Every subset $S \subseteq M$ is a metric space in its own right, by reusing the distance function on M. We say that S is a **subspace** of M.

For example, we saw that the circle S^1 is just a subspace of $\mathbb{R}^2$.

It thus becomes important to distinguish between

(i) **"absolute" adjectives** like "complete" or "bounded", which can be applied to both spaces, and hence even to subsets of spaces (by taking a subspace), and

(ii) **"relative" adjectives** like "open (in M)" and "closed (in M)", which make sense only relative to a space, even though people are often sloppy and omit them.

So "$[0,1]$ is complete" makes sense, as does "$[0,1]$ is a complete subset of $\mathbb{R}$", which we take to mean "$[0,1]$ is a complete subspace of $\mathbb{R}$". This is since "complete" is an absolute adjective.

But here are some examples of ways in which relative adjectives require a little more care:

- Consider the sequence 1, 1.4, 1.41, 1.414, ... Viewed as a sequence in $\mathbb{R}$, it converges to $\sqrt{2}$. But if viewed as a sequence in $\mathbb{Q}$, this sequence does *not* converge! Similarly, the sequence 0.9, 0.99, 0.999, 0.9999 does not converge in the space $(0,1)$, although it does converge in $[0,1]$.

  The fact that these sequences fail to converge even though they "ought to" is weird and bad, and was why we defined complete spaces to begin with.

- In general, it makes no sense to ask a question like "is $[0,1]$ open?". The questions "is $[0,1]$ open in $\mathbb{R}$?" and "is $[0,1]$ open in $[0,1]$?" do make sense, however. The answer to the first question is "no" but the answer to the second question is "yes"; indeed, every space is open in itself. Similarly, $[0, \frac{1}{2})$ is an open set in the space $M = [0,1]$ because it is the ball *in M* of radius $\frac{1}{2}$ centered at 0.

- Dually, it doesn't make sense to ask "is $[0,1]$ closed"? It is closed *in* $\mathbb{R}$ and *in itself* (but every space is closed in itself, anyways).

To make sure you understand the above, here are two exercises to help you practice relative adjectives.

Exercise 6.4.2. Let M be a complete metric space and let $S \subseteq M$. Prove that S is complete if and only if it is closed in M. In particular, $[0,1]$ is complete.

Exercise 6.4.3. Let $M = [0,1] \cup (2,3)$. Show that $[0,1]$ and $(2,3)$ are both open and closed in M.

This illustrates a third point: a nontrivial set can be both open and closed.[1] As we'll see in Chapter 7, this implies the space is disconnected; i.e. the only examples look quite like the one we've given above.

§6.5 A few harder problems to think about

Problem 6A† (Banach fixed point theorem)**.** Let $M = (M, d)$ be a complete metric space. Suppose $T\colon M \to M$ is a continuous map such that for any $p, q \in M$,

$$d\left(T(p), T(q)\right) \le 0.999 d(p, q).$$

(We call T a **contraction**.) Show that T has a unique fixed point.

Problem 6B (Henning Makholm, on math.SE)**.** We let M and N denote the metric spaces obtained by equipping $\mathbb{R}$ with the following two metrics:

$$d_M(x, y) = \min\{1, |x - y|\}$$
$$d_N(x, y) = |e^x - e^y|.$$

(a) Fill in the following 2×3 table with "yes" or "no" for each cell.

	Complete?	Bounded?	Totally bounded?
M			
N			

(b) Are M and N homeomorphic?

Problem 6C† (Completion of a metric space)**.** Let M be a metric space. Construct a complete metric space $\overline{M}$ such that M is a subspace of $\overline{M}$, and every open set of $\overline{M}$ contains a point of M (meaning M is **dense** in $\overline{M}$).

Problem 6D. Show that a metric space is totally bounded if and only if any sequence has a Cauchy subsequence.

Problem 6E. Prove that $\mathbb{Q}$ is not homeomorphic to any complete metric space.

[1]Which always gets made fun of.

7 Topological spaces

In Chapter 2 we introduced the notion of space by describing metrics on them. This gives you a lot of examples, and nice intuition, and tells you how you should draw pictures of open and closed sets.

However, moving forward, it will be useful to begin thinking about topological spaces in terms of just their *open sets*. (One motivation is that our fishy Example 6.3.1 shows that in some ways the notion of homeomorphism really wants to be phrased in terms of open sets, not in terms of the metric.) As we are going to see, the open sets manage to actually retain nearly all the information we need, but are simpler.[1] This will be done in just a few sections, and after that we will start describing more adjectives that we can apply to topological (and hence metric) spaces.

The most important topological notion is missing from this chapter: that of a *compact* space. It is so important that I have dedicated a separate chapter just for it.

Quick note for those who care: the adjectives "Hausdorff", "connected", and later "compact" are all absolute adjectives.

§7.1 Forgetting the metric

Recall Theorem 2.6.11:

> A function $f\colon M \to N$ of metric spaces is continuous if and only if the pre-image of every open set in N is open in M.

Despite us having defined this in the context of metric spaces, this nicely doesn't refer to the metric at all, only the open sets. As alluded to at the start of this chapter, this is a great motivation for how we can forget about the fact that we had a metric to begin with, and rather *start* with the open sets instead.

Definition 7.1.1. A **topological space** is a pair $(X, \mathcal{T})$, where X is a set of points, and $\mathcal{T}$ is the **topology**, which consists of several subsets of X, called the **open sets** of X. The topology must obey the following axioms.

- $\varnothing$ and X are both in $\mathcal{T}$.
- Finite intersections of open sets are also in $\mathcal{T}$.
- Arbitrary unions (possibly infinite) of open sets are also in $\mathcal{T}$.

So this time, the open sets are *given*. Rather than defining a metric and getting open sets from the metric, we instead start from just the open sets.

Abuse of Notation 7.1.2. We abbreviate $(X, \mathcal{T})$ by just X, leaving the topology $\mathcal{T}$ implicit. (Do you see a pattern here?)

[1] The reason I adamantly introduce metric spaces first is because I think otherwise the examples make much less sense.

Example 7.1.3 (Examples of topologies)

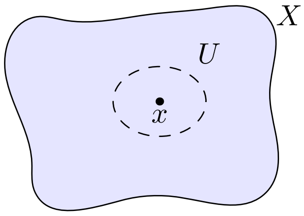
(a) Given a metric space M, we can let $\mathcal{T}$ be the open sets in the metric sense. The point is that the axioms are satisfied.

(b) In particular, **discrete space** is a topological space in which every set is open. (Why?)

(c) Given X, we can let $\mathcal{T} = \{\varnothing, X\}$, the opposite extreme of the discrete space.

Now we can port over our metric definitions.

Definition 7.1.4. An **open neighborhood**[2] of a point $x \in X$ is an open set U which contains x (see figure).



Abuse of Notation 7.1.5. Just to be perfectly clear: by an "open neighborhood" I mean *any* open set containing x. But by an "r-neighborhood" I always mean the points with distance less than r from x, and so I can only use this term if my space is a metric space.

§7.2 Re-definitions

Now that we've defined a topological space, for nearly all of our metric notions we can write down as the definition the one that required only open sets (which will of course agree with our old definitions when we have a metric space).

§7.2.i Continuity

Here was our motivating example, continuity:

Definition 7.2.1. We say function $f\colon X \to Y$ of topological spaces is **continuous** at a point $p \in X$ if the pre-image of any open neighborhood of $f(p)$ is an open neighborhood of p. The function is continuous if it is continuous at every point.

Thus homeomorphism carries over: a bijection which is continuous in both directions.

Definition 7.2.2. A **homeomorphism** of topological spaces (X, τ_X) and (Y, τ_Y) is a bijection $f\colon X \to Y$ which induces a bijection from τ_X to τ_Y: i.e. the bijection preserves open sets.

Question 7.2.3. Show that this is equivalent to f and its inverse both being continuous.

Therefore, any property defined only in terms of open sets is preserved by homeomorphism. Such a property is called a **topological property**. The later adjectives we define ("connected", "Hausdorff", "compact") will all be defined only in terms of open sets, so they will be topological properties.

[2] In literature, a "neighborhood" refers to a set which contains some open set around x. We will not use this term, and exclusively refer to "open neighborhoods".

§7.2.ii Closed sets

We saw last time there were two equivalent definitions for closed sets, but one of them relies only on open sets, and we use it:

Definition 7.2.4. In a general topological space X, we say that $S \subseteq X$ is **closed** in X if the complement $X \setminus S$ is open in X.

If $S \subseteq X$ is any set, the **closure** of S, denoted $\overline{S}$, is defined as the smallest closed set containing S.

Thus for general topological spaces, open and closed sets carry the same information, and it is entirely a matter of taste whether we define everything in terms of open sets or closed sets. In particular, you can translate axioms and properties of open sets to closed ones:

Question 7.2.5. Show that the (possibly infinite) intersection of closed sets is closed while the union of finitely many closed sets is closed. (Look at complements.)

Exercise 7.2.6. Show that a function is continuous if and only if the pre-image of every closed set is closed.

Mathematicians seem to have agreed that they like open sets better.

§7.2.iii Properties that don't carry over

Not everything works:

Remark 7.2.7 (Complete and (totally) bounded are metric properties) — The two metric properties we have seen, "complete" and "(totally) bounded", are not topological properties. They rely on a metric, so as written we cannot apply them to topological spaces. One might hope that maybe, there is some alternate definition (like we saw for "continuous function") that is just open-set based. But Example 6.3.1 showing $(0,1) \cong \mathbb{R}$ tells us that it is hopeless.

Remark 7.2.8 (Sequences don't work well) — You could also try to port over the notion of sequences and convergent sequences. However, this turns out to break a lot of desirable properties. Therefore I won't bother to do so, and thus if we are discussing sequences you should assume that we are working with a metric space.

§7.3 Hausdorff spaces

Prototypical example for this section: Every space that's not the Zariski topology (defined much later).

As you might have guessed, there exist topological spaces which cannot be realized as metric spaces (in other words, are not **metrizable**). One example is just to take $X = \{a, b, c\}$ and the topology $\tau_X = \{\varnothing, \{a, b, c\}\}$. This topology is fairly "stupid": it can't tell apart any of the points a, b, c! But any metric space can tell its points apart (because $d(x, y) > 0$ when $x \neq y$).

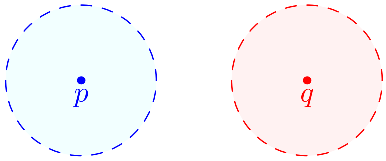
We'll see less trivial examples later, but for now we want to add a little more sanity condition onto our spaces. There is a whole hierarchy of such axioms, labelled T_n for integers n (with $n = 0$ being the weakest and $n = 6$ the strongest); these axioms are called **separation axioms**.

By far the most common hypothesis is the T_2 axiom, which bears a special name.

Definition 7.3.1. A topological space X is **Hausdorff** if for any two distinct points p and q in X, there exists an open neighborhood U of p and an open neighborhood V of q such that

$$U \cap V = \varnothing.$$

In other words, around any two distinct points we should be able to draw disjoint open neighborhoods. Here's a picture to go with above, but not much going on.



Question 7.3.2. Show that all metric spaces are Hausdorff.

I just want to define this here so that I can use this word later. In any case, basically any space we will encounter other than the Zariski topology is Hausdorff.

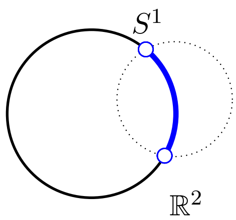
§7.4 Subspaces

Prototypical example for this section: S^1 is a subspace of $\mathbb{R}^2$.

One can also take subspaces of general topological spaces.

Definition 7.4.1. Given a topological space X, and a subset $S \subseteq X$, we can make S into a topological space by declaring that the open subsets of S are $U \cap S$ for open $U \subseteq X$. This is called the **subspace topology**.

So for example, if we view S^1 as a subspace of $\mathbb{R}^2$, then any open arc is an open set, because you can view it as the intersection of an open disk with S^1.



Needless to say, for metric spaces it doesn't matter which of these definitions I choose. (Proving this turns out to be surprisingly annoying, so I won't do so.)

§7.5 Connected spaces

Prototypical example for this section: $[0,1] \cup [2,3]$ is disconnected.

Even in metric spaces, it is possible for a set to be both open and closed.

Definition 7.5.1. A subset S of a topological space X is **clopen** if it is both closed and open in X. (Equivalently, both S and its complement are open.)

For example $\varnothing$ and the entire space are examples of clopen sets. In fact, the presence of a nontrivial clopen set other than these two leads to a so-called *disconnected* space.

Question 7.5.2. Show that a space X has a nontrivial clopen set (one other than $\varnothing$ and X) if and only if X can be written as a disjoint union of two nonempty open sets.

We say X is **disconnected** if there are nontrivial clopen sets, and **connected** otherwise. To see why this should be a reasonable definition, it might help to solve Problem 7A†.

Example 7.5.3 (Disconnected and connected spaces)

(a) The metric space

$$\{(x,y) \mid x^2+y^2 \le 1\} \cup \{(x,y) \mid (x-4)^2+y^2 \le 1\} \subseteq \mathbb{R}^2$$

is disconnected (it consists of two disks).

(b) The space $[0,1] \cup [2,3]$ is disconnected: it consists of two segments, each of which is a clopen set.

(c) A discrete space on more than one point is disconnected, since *every* set is clopen in the discrete space.

(d) Convince yourself that the set

$$\left\{x \in \mathbb{Q} \mid x^2 < 2014\right\}$$

is a clopen subset of $\mathbb{Q}$. Hence $\mathbb{Q}$ is disconnected too – it has *gaps*.

(e) $[0,1]$ is connected.

§7.6 Path-connected spaces

Prototypical example for this section: Walking around in $\mathbb{C}$.

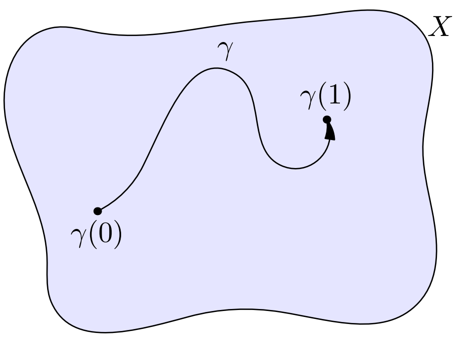
A stronger and perhaps more intuitive notion of a connected space is a *path-connected* space. The short description: "walk around in the space".

Definition 7.6.1. A **path** in the space X is a continuous function

$$\gamma\colon [0,1] \to X.$$

Its **endpoints** are the two points $\gamma(0)$ and $\gamma(1)$.

You can think of $[0,1]$ as measuring "time", and so we'll often write $\gamma(t)$ for $t \in [0,1]$ (with t standing for "time"). Here's a picture of a path.



Question 7.6.2. Why does this agree with your intuitive notion of what a "path" is?

Definition 7.6.3. A space X is **path-connected** if any two points in it are connected by some path.

Exercise 7.6.4 (Path-connected implies connected)**.** Let $X = U \sqcup V$ be a disconnected space. Show that there is no path from a point of U to point V. (If $\gamma\colon [0,1] \to X$, then we get $[0,1] = \gamma^{\text{pre}}(U) \sqcup \gamma^{\text{pre}}(V)$, but $[0,1]$ is connected.)

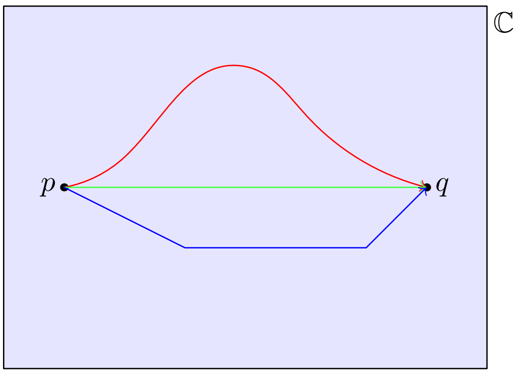
Example 7.6.5 (Examples of path-connected spaces)

- $\mathbb{R}^2$ is path-connected, since we can "connect" any two points with a straight line.
- The unit circle S^1 is path-connected, since we can just draw the major or minor arc to connect two points.

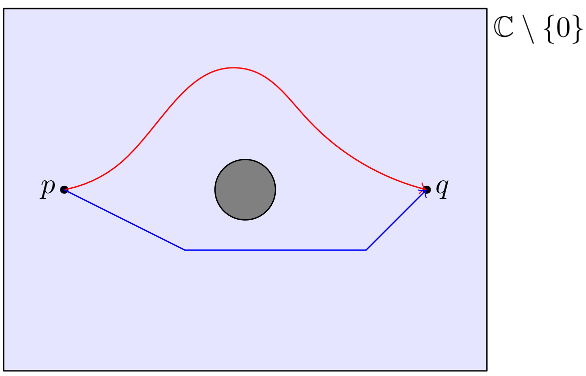
§7.7 Homotopy and simply connected spaces

Prototypical example for this section: $\mathbb{C}$ *and* $\mathbb{C} \setminus \{0\}$.

Now let's motivate the idea of homotopy. Consider the example of the complex plane $\mathbb{C}$ (which you can think of just as $\mathbb{R}^2$) with two points p and q. There's a whole bunch of paths from p to q but somehow they're not very different from one another. If I told you "walk from p to q" you wouldn't have too many questions.



So we're living happily in $\mathbb{C}$ until a meteor strikes the origin, blowing it out of existence. Then suddenly to get from p to q, people might tell you two different things: "go left around the meteor" or "go right around the meteor".



So what's happening? In the first picture, the red, green, and blue paths somehow all looked the same: if you imagine them as pieces of elastic string pinned down at p and q, you can stretch each one to any other one.

But in the second picture, you can't move the red string to match with the blue string: there's a meteor in the way. The paths are actually different.[3]

The formal notion we'll use to capture this is *homotopy equivalence*. We want to write a definition such that in the first picture, the three paths are all *homotopic*, but the two paths in the second picture are somehow not homotopic. And the idea is just continuous deformation.

Definition 7.7.1. Let α and β be paths in X whose endpoints coincide. A (path) **homotopy** from α to β is a continuous function $F\colon [0,1]^2 \to X$, which we'll write $F_s(t)$ for $s, t \in [0,1]$, such that

$$F_0(t) = \alpha(t) \text{ and } F_1(t) = \beta(t) \text{ for all } t \in [0,1]$$

and moreover

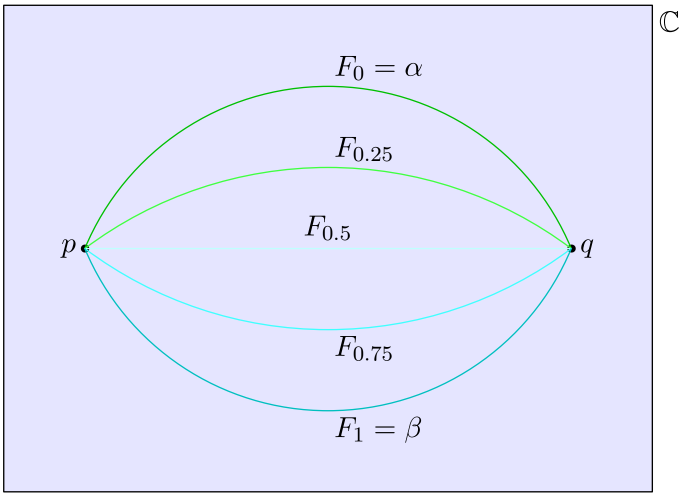
$$\alpha(0) = \beta(0) = F_s(0) \text{ and } \alpha(1) = \beta(1) = F_s(1) \text{ for all } s \in [0,1].$$

If a path homotopy exists, we say α and β are path **homotopic** and write $\alpha \simeq \beta$.

Abuse of Notation 7.7.2. While I strictly should say "path homotopy" to describe this relation between two paths, I will shorten this to just "homotopy" instead. Similarly I will shorten "path homotopic" to "homotopic".

Animated picture: `https://commons.wikimedia.org/wiki/File:HomotopySmall.gif`. Needless to say, $\simeq$ is an equivalence relation.

What this definition is doing is taking α and "continuously deforming" it to β, while keeping the endpoints fixed. Note that for each particular s, F_s is itself a function. So s represents time as we deform α to β: it goes from 0 to 1, starting at α and ending at β.



> **Question 7.7.3.** Convince yourself the above definition is right. What goes wrong when the meteor strikes?

So now I can tell you what makes $\mathbb{C}$ special:

[3] If you know about winding numbers, you might feel this is familiar. We'll talk more about this in the chapter on the fundamental group.

Definition 7.7.4. A space X is **simply connected** if it's path-connected and for any points p and q, all paths from p to q are homotopic.

That's why you don't ask questions when walking from p to q in $\mathbb{C}$: there's really only one way to walk. Hence the term "simply" connected.

> **Question 7.7.5.** Convince yourself that $\mathbb{R}^n$ is simply connected for all n.

§7.8 Bases of spaces

Prototypical example for this section: *$\mathbb{R}$ has a basis of open intervals, and $\mathbb{R}^2$ has a basis of open disks.*

You might have noticed that the open sets of $\mathbb{R}$ are a little annoying to describe: the prototypical example of an open set is $(0,1)$, but there are other open sets like

$$(0,1) \cup \left(1, \frac{3}{2}\right) \cup \left(2, \frac{7}{3}\right) \cup (2014, 2015).$$

> **Question 7.8.1.** Check this is an open set.

But okay, this isn't *that* different. All I've done is taken a bunch of my prototypes and threw a bunch of $\cup$ signs at it. And that's the idea behind a basis.

Definition 7.8.2. A **basis** for a topological space X is a subset $\mathcal{B}$ of the open sets such that every open set in X is a union of some (possibly infinite) number of elements in $\mathcal{B}$.

And all we're doing is saying:

> **Example 7.8.3** (Basis of $\mathbb{R}$)
> The open intervals form a basis of $\mathbb{R}$.

In fact, more generally we have:

> **Theorem 7.8.4** (Basis of metric spaces)
> The r-neighborhoods form a basis of any metric space M.

Proof. Kind of silly – given an open set U, for every point p inside U, draw an r_p-neighborhood U_p contained entirely inside U. Then $\bigcup_p U_p$ is contained in U and covers every point inside it. $\square$

Hence, an open set in $\mathbb{R}^2$ is nothing more than a union of a bunch of open disks, and so on. The point is that in a metric space, the only open sets you really ever have to worry too much about are the r-neighborhoods.

§7.9 A few harder problems to think about

Problem 7A†. Let X be a topological space. Show that there exists a nonconstant continuous function $X \to \{0,1\}$ if and only if X is disconnected (here $\{0,1\}$ is given the discrete topology).

Problem 7B*. Let X and Y be topological spaces and let $f: X \to Y$ be a continuous function.

(a) Show that if X is connected then so is $f^{\text{img}}(X)$.

(b) Show that if X is path-connected then so is $f^{\text{img}}(X)$.

Problem 7C (Hausdorff implies T_1 axiom)**.** Let X be a Hausdorff topological space. Prove that for any point $p \in X$ the set $\{p\}$ is closed.

Problem 7D ([**Pu02**], Exercise 2.56)**.** Let M be a metric space with more than one point but at most countably infinitely many points. Show that M is disconnected.

Problem 7E. Let X be a topological space. The *connected component* of a point $p \in X$ is the union of all subspaces $S \subseteq X$ which are connected and contain p.

(a) Does the connected component of a point have to be itself connected?

(b) Does the connected component of a point have to be an open subset of X?

Problem 7F (Furstenberg)**.** We declare a subset of $\mathbb{Z}$ to be open if it's the union (possibly empty or infinite) of arithmetic sequences $\{a + nd \mid n \in \mathbb{Z}\}$, where a and d are positive integers.

(a) Verify this forms a topology on $\mathbb{Z}$, called the **evenly spaced integer topology**.

(b) Prove there are infinitely many primes by considering $\bigcup_p p\mathbb{Z}$ for primes p.

Problem 7G. Prove that the evenly spaced integer topology on $\mathbb{Z}$ is metrizable. In other words, show that one can impose a metric $d: \mathbb{Z}^2 \to \mathbb{R}$ which makes $\mathbb{Z}$ into a metric space whose open sets are those described above.

Problem 7H. We know that any open set $U \subseteq \mathbb{R}$ is a union of open intervals (allowing $\pm\infty$ as endpoints). One can show that it's actually possible to write U as the union of *pairwise disjoint* open intervals.[4] Prove that there exists such a disjoint union with at most *countably many* intervals in it.

[4]You are invited to try and prove this, but I personally found the proof quite boring.

8 Compactness

One of the most important notions of topological spaces is that of *compactness*. It generalizes the notion of "closed and bounded" in Euclidean space to any topological space (e.g. see Problem 8F†).

For metric spaces, there are two equivalent ways of formulating compactness:

- A "natural" definition using *sequences*, called sequential compactness.
- A less natural definition using open covers.

As I alluded to earlier, sequences in metric spaces are super nice, but sequences in general topological spaces *suck* (to the point where I didn't bother to define convergence of general sequences). So it's the second definition that will be used for general spaces.

§8.1 Definition of sequential compactness

Prototypical example for this section: $[0, 1]$ *is compact, but* $(0, 1)$ *is not.*

To emphasize, compactness is one of the *best* possible properties that a metric space can have.

Definition 8.1.1. A **subsequence** of an infinite sequence $x_1, x_2, \dots$ is exactly what it sounds like: a sequence $x_{i_1}, x_{i_2}, \dots$ where $i_1 < i_2 < \cdots$ are positive integers. Note that the sequence is required to be infinite.

Another way to think about this is "selecting infinitely many terms" or "deleting some terms" of the sequence, depending on whether your glass is half empty or half full.

Definition 8.1.2. A metric space M is **sequentially compact** if every sequence has a subsequence which converges.

This time, let me give some non-examples before the examples.

Example 8.1.3 (Non-examples of compact metric spaces)

(a) The space $\mathbb{R}$ is not compact: consider the sequence $1, 2, 3, 4, \dots$. Any subsequence explodes, hence $\mathbb{R}$ cannot possibly be compact.

(b) More generally, if a space is not bounded it cannot be compact. (You can prove this if you want.)

(c) The open interval $(0, 1)$ is bounded but not compact: consider the sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$. No subsequence can converge to a point in $(0, 1)$ because the sequence "converges to 0".

(d) More generally, any space which is not complete cannot be compact.

Now for the examples!

Question 8.1.4. Show that a finite set is compact. (Pigeonhole Principle.)

Example 8.1.5 (Examples of compact spaces)

Here are some more examples of compact spaces. I'll prove they're compact in just a moment; for now just convince yourself they are.

(a) $[0, 1]$ is compact. Convince yourself of this! Imagine having a large number of dots in the unit interval…

(b) The surface of a sphere, $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ is compact.

(c) The unit ball $B^2 = \{(x, y) \mid x^2 + y^2 \le 1\}$ is compact.

(d) The **Hawaiian earring** living in $\mathbb{R}^2$ is compact: it consists of mutually tangent circles of radius $\frac{1}{n}$ for each n, as in Figure 8.1.

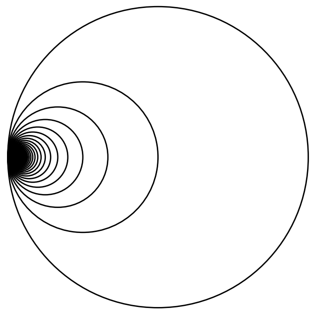


Figure 8.1: Hawaiian Earring.

To aid in generating more examples, we remark:

Proposition 8.1.6 (Closed subsets of compacts)

Closed subsets of sequentially compact sets are compact.

Question 8.1.7. Prove this. (It should follow easily from definitions.)

We need to do a bit more work for these examples, which we do in the next section.

§8.2 Criteria for compactness

Theorem 8.2.1 (Tychonoff's theorem)

If X and Y are compact spaces, then so is $X \times Y$.

Proof. Problem 8E. □

We also have:

Theorem 8.2.2 (The interval is compact)

$[0,1]$ is compact.

Proof. Killed by Problem 8F†; however, here is a sketch of a direct proof. Split $[0,1]$ into $[0,\frac{1}{2}] \cup [\frac{1}{2},1]$. By Pigeonhole, infinitely many terms of the sequence lie in the left half (say); let x_1 be the first one and then keep only the terms in the left half after x_1. Now split $[0,\frac{1}{2}]$ into $[0,\frac{1}{4}] \cup [\frac{1}{4},\frac{1}{2}]$. Again, by Pigeonhole, infinitely many terms fall in some half; pick one of them, call it x_2. Rinse and repeat. In this way we generate a sequence x_1, x_2, ... which is Cauchy, implying that it converges since $[0,1]$ is complete. □

Now we can prove the main theorem about Euclidean space: in $\mathbb{R}^n$, compactness is equivalent to being "closed and bounded".

Theorem 8.2.3 (Bolzano-Weierstraß)

A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Question 8.2.4. Why does this imply the spaces in our examples are compact?

Proof. Well, look at a closed and bounded $S \subseteq \mathbb{R}^n$. Since it's bounded, it lives inside some box $[a_1,b_1] \times [a_2,b_2] \times \cdots \times [a_n,b_n]$. By Tychonoff's theorem, since each $[a_i,b_i]$ is compact the entire box is. Since S is a closed subset of this compact box, we're done. □

One really has to work in $\mathbb{R}^n$ for this to be true! In other spaces, this criterion can easily fail.

Example 8.2.5 (Closed and bounded but not compact)

Let $S = \{s_1, s_2, \dots\}$ be any infinite set equipped with the discrete metric. Then S is closed (since all convergent sequences are constant sequences) and S is bounded (all points are a distance 1 from each other) but it's certainly not compact since the sequence $s_1, s_2, \dots$ doesn't converge.

The Bolzano-Weierstrass theorem, which is Problem 8F†, tells you exactly which sets are compact in metric spaces in a geometric way.

§8.3 Compactness using open covers

Prototypical example for this section: $[0,1]$ *is compact.*

There's a second related notion of compactness which I'll now define. The following definitions might appear very unmotivated, but bear with me.

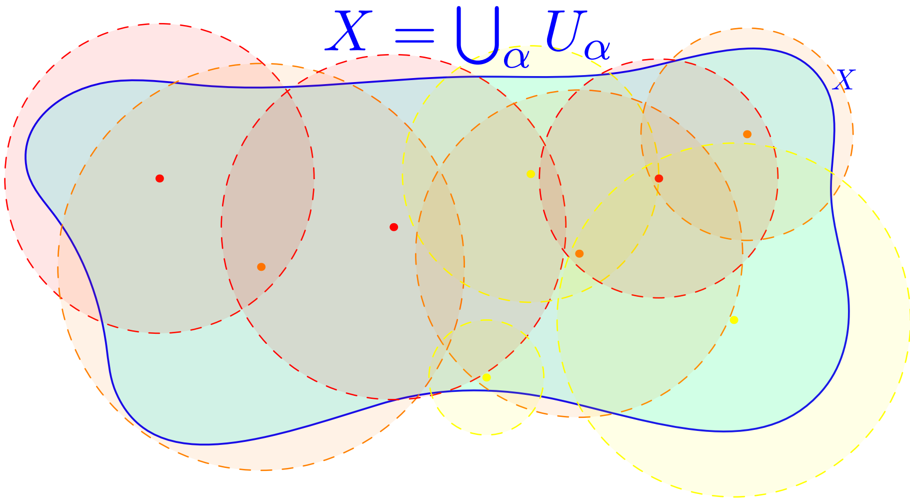
Definition 8.3.1. An open cover of a topological space X is a collection of open sets $\{U_\alpha\}$ (possibly infinite or uncountable) which *cover* it: every point in X lies in at least one of the U_α, so that

$$X = \bigcup U_\alpha.$$

Such a cover is called an **open cover**.

A **subcover** is exactly what it sounds like: it takes only some of the U_α, while ensuring that X remains covered.

Some art:



Definition 8.3.2. A topological space X is **quasicompact** if *every* open cover has a finite subcover. It is **compact** if it is also Hausdorff.

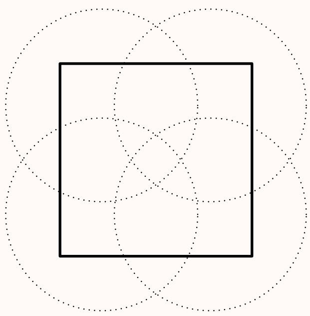
Remark 8.3.3 — The "Hausdorff" hypothesis that I snuck in is a sanity condition which is not worth worrying about unless you're working on the algebraic geometry chapters, since all the spaces you will deal with are Hausdorff. (In fact, some authors don't even bother to include it.) For example all metric spaces are Hausdorff and thus this condition can be safely ignored if you are working with metric spaces.

What does this mean? Here's an example:

Example 8.3.4 (Example of a finite subcover)

Suppose we cover the unit square $M = [0,1]^2$ by putting an open disk of diameter 1 centered at every point (trimming any overflow). This is clearly an open cover because, well, every point lies in *many* of the open sets, and in particular is the center of one.

But this is way overkill – we only need about four of these circles to cover the whole square. That's what is meant by a "finite subcover".



Why do we care? Because of this:

Theorem 8.3.5 (Sequentially compact $\iff$ compact)

A metric space M is sequentially compact if and only if it is compact.

We defer the proof to the last section.

This gives us the motivation we wanted for our definition. Sequential compactness was a condition that made sense. The open-cover definition looked strange, but it turned out to be equivalent. But we now prefer it, because we have seen that whenever possible we want to resort to open-set-only based definitions: so that e.g. they are preserved under homeomorphism.

Example 8.3.6 (An example of non-compactness)

The space $X = [0,1)$ is not compact in either sense. We can already see it is not sequentially compact, because it is not even complete (look at $x_n = 1 - \frac{1}{n}$). To see it is not compact under the covering definition, consider the sets

$$U_m = \left[0, 1 - \frac{1}{m+1}\right)$$

for $m = 1, 2, \dots$. Then $X = \bigcup U_i$; hence the U_i are indeed a cover. But no finite collection of the U_i's will cover X.

Question 8.3.7. Convince yourself that $[0,1]$ *is* compact; this is a little less intuitive than it being sequentially compact.

Abuse of Notation 8.3.8. Thus, we'll never call a metric space "sequentially compact" again — we'll just say "compact". (Indeed, I kind of already did this in the previous few sections.)

§8.4 Applications of compactness

Compactness lets us reduce *infinite* open covers to finite ones. Actually, it lets us do this even if the open covers are *blithely stupid*. Very often one takes an open cover consisting of an open neighborhood of $x \in X$ for every single point x in the space; this is a huge number of open sets, and yet compactness lets us reduce to a finite set.

To give an example of a typical usage:

Proposition 8.4.1 (Compact $\implies$ totally bounded)

Let M be compact. Then M is totally bounded.

Proof using covers. For every point $p \in M$, take an ε-neighborhood of p, say U_p. These cover M for the horrendously stupid reason that each point p is at the very least covered by its open neighborhood U_p. Compactness then lets us take a finite subcover. □

Next, an important result about maps between compact spaces.

Theorem 8.4.2 (Images of compacts are compact)

Let $f\colon X \to Y$ be a continuous function, where X is compact. Then the image

$$f^{\text{img}}(X) \subseteq Y$$

is compact.

Proof using covers. Take any open cover $\{V_\alpha\}$ in Y of $f^{\text{img}}(X)$. By continuity of f, it pulls back to an open cover $\{U_\alpha\}$ of X. Thus some finite subcover of this covers X. The corresponding V's cover $f^{\text{img}}(X)$. $\square$

> **Question 8.4.3.** Give another proof using the sequential definitions of continuity and compactness. (This is even easier.)

Some nice corollaries of this:

Corollary 8.4.4 (Extreme value theorem)

Let X be compact and consider a continuous function $f\colon X \to \mathbb{R}$. Then f achieves a *maximum value* at some point, i.e. there is a point $p \in X$ such that $f(p) \ge f(q)$ for any other $q \in X$.

Corollary 8.4.5 (Intermediate value theorem)

Consider a continuous function $f\colon [0,1] \to \mathbb{R}$. Then the image of f is of the form $[a,b]$ for some real numbers $a \le b$.

Sketch of Proof. The point is that the image of f is compact in $\mathbb{R}$, and hence closed and bounded. You can convince yourself that the closed sets are just unions of closed intervals. That implies the extreme value theorem.

When $X = [0,1]$, the image is also connected, so there should only be one closed interval in $f^{\text{img}}([0,1])$. Since the image is bounded, we then know it's of the form $[a,b]$. (To give a full proof, you would use the so-called *least upper bound* property, but that's a little involved for a bedtime story; also, I think $\mathbb{R}$ is boring.) $\square$

Example 8.4.6 ($1/x$)

The compactness hypothesis is really important here. Otherwise, consider the function

$$(0,1) \to \mathbb{R} \quad \text{by} \quad x \mapsto \frac{1}{x}.$$

This function (which you plot as a hyperbola) is not bounded; essentially, you can see graphically that the issue is we can't extend it to a function on $[0,1]$ because it explodes near $x = 0$.

One last application: if M is a compact metric space, then continuous functions $f\colon M \to N$ are continuous in an especially "nice" way:

Definition 8.4.7. A function $f\colon M \to N$ of metric spaces is called **uniformly continuous** if for any $\varepsilon > 0$, there exists a $\delta > 0$ (depending only on ε) such that whenever $d_M(x,y) < \delta$ we also have $d_N(f(x), f(y)) < \varepsilon$.

The name means that for $\varepsilon > 0$, we need a δ that works for *every point* of M.

Example 8.4.8 (Uniform continuity)

(a) The functions $\mathbb{R}$ to $\mathbb{R}$ of the form $x \mapsto ax+b$ are all uniformly continuous, since one can always take $\delta = \varepsilon/|a|$ (or $\delta = 1$ if $a = 0$).

(b) Actually, it is true that a differentiable function $\mathbb{R} \to \mathbb{R}$ with a bounded derivative is uniformly continuous. (The converse is false for the reason that uniformly continuous doesn't imply differentiable at all.)

(c) The function $f \colon \mathbb{R} \to \mathbb{R}$ by $x \mapsto x^2$ is *not* uniformly continuous, since for large x, tiny δ changes to x lead to fairly large changes in x^2. (If you like, you can try to prove this formally now.)

Think $f(2017.01) - f(2017) > 40$; even when $\delta = 0.01$, one can still cause large changes in f.

(d) However, when restricted to $(0,1)$ or $[0,1]$ the function $x \mapsto x^2$ becomes uniformly continuous. (For $\varepsilon > 0$ one can now pick for example $\delta = \min\{1, \varepsilon\}/3$.)

(e) The function $(0,1) \to \mathbb{R}$ by $x \mapsto 1/x$ is *not* uniformly continuous (same reason as before).

Now, as promised:

Proposition 8.4.9 (Continuous on compact $\implies$ uniformly continuous)

If M is compact and $f \colon M \to N$ is continuous, then f is uniformly continuous.

Proof using sequences. Fix $\varepsilon > 0$, and assume for contradiction that for every $\delta = 1/k$ there exists points x_k and y_k within δ of each other but with images $\varepsilon > 0$ apart. By compactness, take a convergent subsequence $x_{i_k} \to p$. Then $y_{i_k} \to p$ as well, since the x_k's and y_k's are close to each other. So both sequences $f(x_{i_k})$ and $f(y_{i_k})$ should converge to $f(p)$ by sequential continuity, but this can't be true since the two sequences are always ε apart. □

§8.5 (Optional) Equivalence of formulations of compactness

We will prove that:

Theorem 8.5.1 (Heine-Borel for general metric spaces)

For a metric space M, the following are equivalent:

(i) Every sequence has a convergent subsequence,

(ii) The space M is complete and totally bounded, and

(iii) Every open cover has a finite subcover.

We leave the proof that (i) $\iff$ (ii) as Problem 8F†; the idea of the proof is much in the spirit of Theorem 8.2.2.

Proof that (i) and (ii) $\implies$ (iii). We prove the following lemma, which is interesting in its own right.

Lemma 8.5.2 (Lebesgue number lemma)

Let M be a compact metric space and $\{U_\alpha\}$ an open cover. Then there exists a real number $\delta > 0$, called a **Lebesgue number** for that covering, such that the δ-neighborhood of any point p lies entirely in some U_α.

Proof of lemma. Assume for contradiction that for every $\delta = 1/k$ there is a point $x_k \in M$ such that its $1/k$-neighborhood isn't contained in any U_α. In this way we construct a sequence $x_1, x_2, \dots$; thus we're allowed to take a subsequence which converges to some x. Then for every $\varepsilon > 0$ we can find an integer n such that $d(x_n, x) + 1/n < \varepsilon$; thus the ε-neighborhood at x isn't contained in any U_α for every $\varepsilon > 0$. This is impossible, because we assumed x was covered by some open set. ∎

Now, take a Lebesgue number δ for the covering. Since M is totally bounded, finitely many δ-neighborhoods cover the space, so finitely many U_α do as well. □

Proof that (iii) $\implies$ (ii). One step is immediate:

> **Question 8.5.3.** Show that the covering condition $\implies$ totally bounded.

The tricky part is showing M is complete. Assume for contradiction it isn't and thus that the sequence (x_k) is Cauchy, but it doesn't converge to any particular point.

> **Question 8.5.4.** Show that this implies for each $p \in M$, there is an ε_p-neighborhood U_p which contains at most finitely many of the points of the sequence (x_k). (You will have to use the fact that $x_k \not\to p$ and (x_k) is Cauchy.)

Now if we consider $M = \bigcup_p U_p$ we get a finite subcover of these open neighborhoods; but this finite subcover can only cover finitely many points of the sequence, by contradiction. □

§8.6 A few harder problems to think about

The later problems are pretty hard; some have the flavor of IMO 3/6-style constructions. It's important to draw lots of pictures so one can tell what's happening. Of these Problem 8F† is definitely my favorite.

Problem 8A. Show that the closed interval $[0, 1]$ and open interval $(0, 1)$ are not homeomorphic.

Problem 8B. Let X be a topological space with the discrete topology. Under what conditions is X compact?

Problem 8C (The cofinite topology is quasicompact only)**.** We let X be an infinite set and equip it with the **cofinite topology**: the open sets are the empty set and complements of finite sets. This makes X into a topological space. Show that X is quasicompact but not Hausdorff.

Problem 8D (Cantor's intersection theorem)**.** Let X be a compact topological space, and suppose

$$X = K_0 \supseteq K_1 \supseteq K_2 \supseteq \dots$$

is an infinite sequence of nested nonempty closed subsets. Show that $\bigcap_{n \ge 0} K_n \neq \varnothing$.

Problem 8E (Tychonoff's theorem)**.** Let X and Y be compact metric spaces. Show that $X \times Y$ is compact. (This is also true for general topological spaces, but the proof is surprisingly hard, and we haven't even defined $X \times Y$ in general yet.)

Problem 8F† (Bolzano-Weierstraß theorem for general metric spaces)**.** Prove that a metric space M is sequentially compact if and only if it is complete and totally bounded.

Problem 8G (Almost Arzelà-Ascoli theorem)**.** Let $f_1, f_2, \dots : [0,1] \to [-100, 100]$ be an **equicontinuous** sequence of functions, meaning

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall n \, \forall x, y \quad (|x - y| < \delta \implies |f_n(x) - f_n(y)| < \varepsilon)$$

Show that we can extract a subsequence $f_{i_1}, f_{i_2}, \dots$ of these functions such that for every $x \in [0,1]$, the sequence $f_{i_1}(x)$, $f_{i_2}(x)$, $\dots$ converges.

Problem 8H. Let $M = (M, d)$ be a bounded metric space. Suppose that whenever d' is another metric on M for which (M, d) and (M, d') are homeomorphic (i.e. have the same open sets), then d' is also bounded. Prove that M is compact.

Problem 8I. In this problem a "circle" refers to the boundary of a disk with *nonzero* radius.

(a) Is it possible to partition the plane $\mathbb{R}^2$ into disjoint circles?

(b) From the plane $\mathbb{R}^2$ we delete two distinct points p and q. Is it possible to partition the remaining points into disjoint circles?