An Infinitely Large Napkin

https://web.evanchen.cc/napkin.html

Evan Chen

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When introduced to a new idea, always ask why you should care.
Do not expect an answer right away, but demand one eventually.

— *Ravi Vakil* [Va17]

If you like this book and want to support me,
please consider buying me a coffee!

Buy Me a Coffee

https://ko-fi.com/evanchen/

For Brian and Lisa, who finally got me to write it.

This is (still!) an **incomplete draft**. Please send corrections, comments, pictures of kittens, etc. to evan@evanchen.cc, or pull-request at https://github.com/vEnhance/napkin.

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Preface

The origin of the name "Napkin" comes from the following quote of mine.

> I'll be eating a quick lunch with some friends of mine who are still in high school. They'll ask me what I've been up to the last few weeks, and I'll tell them that I've been learning category theory. They'll ask me what category theory is about. I tell them it's about abstracting things by looking at just the structure-preserving morphisms between them, rather than the objects themselves. I'll try to give them the standard example Grp, but then I'll realize that they don't know what a homomorphism is. So then I'll start trying to explain what a homomorphism is, but then I'll remember that they haven't learned what a group is. So then I'll start trying to explain what a group is, but by the time I finish writing the group axioms on my napkin, they've already forgotten why I was talking about groups in the first place. And then it's 1PM, people need to go places, and I can't help but think:
>
> *"Man, if I had forty hours instead of forty minutes, I bet I could actually have explained this all".*

This book was initially my attempt at those forty hours, but has grown considerably since then.

About this book

The *Infinitely Large Napkin* is a light but mostly self-contained introduction to a large amount of higher math.

I should say at once that this book is not intended as a replacement for dedicated books or courses; the amount of depth is not comparable. On the flip side, the benefit of this "light" approach is that it becomes accessible to a larger audience, since the goal is merely to give the reader a feeling for any particular topic rather than to emulate a full semester of lectures.

I initially wrote this book with talented high-school students in mind, particularly those with math-olympiad type backgrounds. Some remnants of that cultural bias can still be felt throughout the book, particularly in assorted challenge problems which are taken from mathematical competitions. However, in general I think this would be a good reference for anyone with some amount of mathematical maturity and curiosity. Examples include but certainly not limited to: math undergraduate majors, physics/CS majors, math PhD students who want to hear a little bit about fields other than their own, advanced high schoolers who like math but not math contests, and unusually intelligent kittens fluent in English.

Source code

The project is hosted on GitHub at https://github.com/vEnhance/napkin. Pull requests are quite welcome! I am also happy to receive suggestions and corrections by email.

Philosophy behind the Napkin approach

As far as I can tell, higher math for high-school students comes in two flavors:

- Someone tells you about the hairy ball theorem in the form "you can't comb the hair on a spherical cat" then doesn't tell you anything about why it should be true, what it means to actually "comb the hair", or any of the underlying theory, leaving you with just some vague notion in your head.
- You take a class and prove every result in full detail, and at some point you stop caring about what the professor is saying.

Presumably you already know how unsatisfying the first approach is. So the second approach seems to be the default, but I really think there should be some sort of middle ground here.

Unlike university, it is *not* the purpose of this book to train you to solve exercises or write proofs,[1] or prepare you for research in the field. Instead I just want to show you some interesting math. The things that are presented should be memorable and worth caring about. For that reason, proofs that would be included for completeness in any ordinary textbook are often omitted here, unless there is some idea in the proof which I think is worth seeing. In particular, I place a strong emphasis over explaining why a theorem *should* be true rather than writing down its proof. This is a recurrent theme of this book:

Natural explanations supersede proofs.

My hope is that after reading any particular chapter in Napkin, one might get the following out of it:

- Knowing what the precise definitions are of the main characters,
- Being acquainted with the few really major examples,
- Knowing the precise statements of famous theorems, and having a sense of why they *should* be true.

Understanding "why" something is true can have many forms. This is sometimes accomplished with a complete rigorous proof; in other cases, it is given by the idea of the proof; in still other cases, it is just a few key examples with extensive cheerleading.

Obviously this is nowhere near enough if you want to e.g. do research in a field; but if you are just a curious outsider, I hope that it's more satisfying than the elevator pitch or Wikipedia articles. Even if you do want to learn a topic with serious depth, I hope that it can be a good zoomed-out overview before you really dive in, because in many senses the choice of material is "what I wish someone had told me before I started".

More pedagogical comments and references

The preface would become too long if I talked about some of my pedagogical decisions chapter by chapter, so Appendix A contains those comments instead.

In particular, I often name specific references, and the end of that appendix has more references. So this is a good place to look if you want further reading.

[1] Which is not to say problem-solving isn't valuable; I myself am a math olympiad coach at heart. It's just not the point of this book.

Historical and personal notes

I began writing this book in December 2014, after having finished my first semester of undergraduate at Harvard. It became my main focus for about 18 months after that, as I became immersed in higher math. I essentially took only math classes (gleefully ignoring all my other graduation requirements), and merged as much of it as I could (as well as lots of other math I learned on my own time) into the Napkin.

Towards August 2016, though, I finally lost steam. The first public drafts went online then, and I decided to step back. Having burnt out slightly, I then took a break from higher math, and spent the remaining two undergraduate years[2] working extensively as a coach for the American math olympiad team, and trying to spend as much time with my friends as I could before they graduated and went their own ways.

During those two years, readers sent me many kind words of gratitude, many reports of errors, and many suggestions for additions. So in November 2018, some weeks into my first semester as a math PhD student, I decided I should finish what I had started. Some months later, here is what I have.

Acknowledgements

I am indebted to countless people for this work. Here is a partial (surely incomplete) list.

- Thanks to all my teachers and professors for teaching me much of the material covered in these notes, as well as the authors of all the references I have cited here. A special call-out to [**Ga14**], [**Le14**], [**Sj05**], [**Ga03**], [**Ll15**], [**Et11**], [**Ko14**], [**Va17**], [**Pu02**], [**Go18**], which were especially influential.
- Thanks also to dozens of friends and strangers who read through preview copies of my draft, and pointed out errors and gave other suggestions. Special mention to Andrej Vuković and Alexander Chua for together catching over a thousand errors. Thanks also to Brian Gu and Tom Tseng for many corrections. (If you find mistakes or have suggestions yourself, I would love to hear them!) Thanks also to Royce Yao and `user202729` for their contributions of guest chapters to the document.
- Thanks to Jenny Chu and Lanie Deng for the cover artwork.
- I'd also like to express my gratitude for many, many kind words I received during the development of this project. These generous comments led me to keep working on this, and were largely responsible for my decision in November 2018 to begin updating the Napkin again.

Finally, a huge thanks to the math olympiad community, from which the Napkin (and me) has its roots. All the enthusiasm, encouragement, and thank-you notes I have received over the years led me to begin writing this in the first place. I otherwise would never have the arrogance to dream a project like this was at all possible. And of course I would be nowhere near where I am today were it not for the life-changing journey I took in chasing my dreams to the IMO. Forever TWN2!

[2] Alternatively: " . . . and spent the next two years forgetting everything I had painstakingly learned". Which made me grateful for all the past notes in the Napkin!

Advice for the reader

§1 Prerequisites

As explained in the preface, the main prerequisite is some amount of mathematical maturity. This means I expect the reader to know how to read and write a proof, follow logical arguments, and so on.

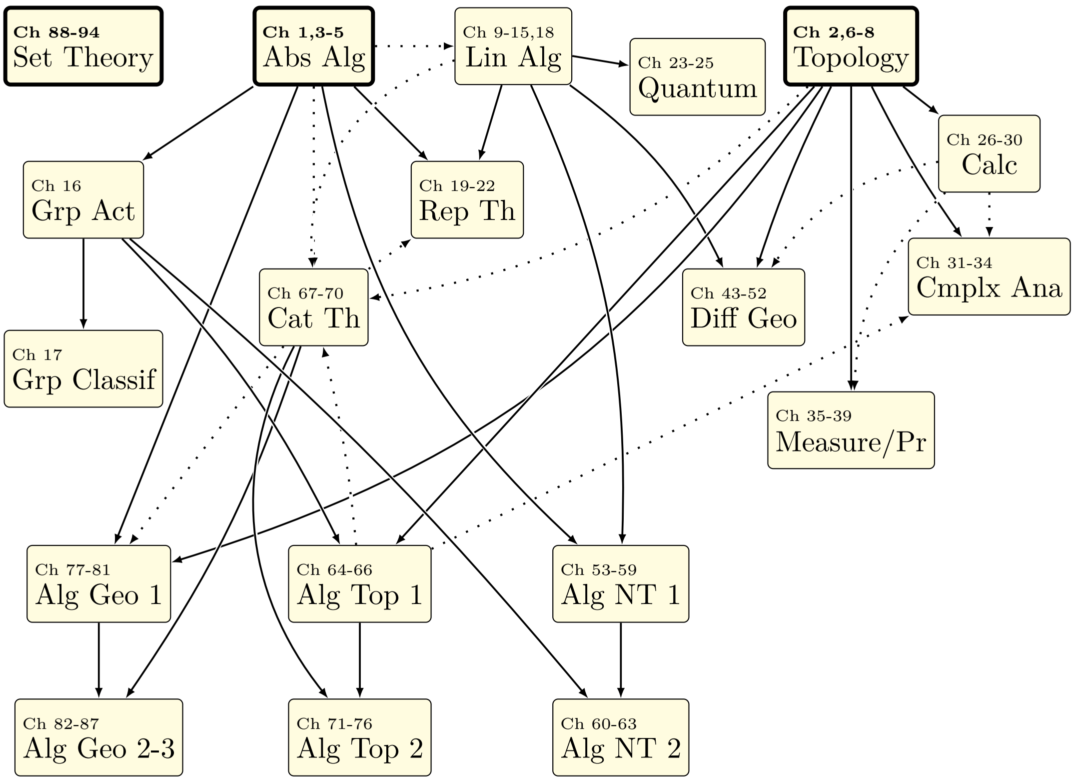
I also assume the reader is familiar with basic terminology about sets and functions (e.g. "what is a bijection?"). If not, one should consult Appendix E.

§2 Deciding what to read

There is no need to read this book in linear order: it covers all sorts of areas in mathematics, and there are many paths you can take. In Chapter 0, I give a short overview for each part explaining what you might expect to see in that part.

For now, here is a brief chart showing how the chapters depend on each other; again see Chapter 0 for details. Dependencies are indicated by arrows; dotted lines are optional dependencies. **I suggest that you simply pick a chapter you find interesting, and then find the shortest path.** With that in mind, I hope the length of the entire PDF is not intimidating.

(The text in the following diagram should be clickable and links to the relevant part.)



§3 Questions, exercises, and problems

In this book, there are three hierarchies:

- An inline **question** is intended to be offensively easy, mostly a chance to help you internalize definitions. If you find yourself unable to answer one or two of them, it probably means I explained it badly and you should complain to me. But if you can't answer many, you likely missed something important: read back.
- An inline **exercise** is more meaty than a question, but shouldn't have any "tricky" steps. Often I leave proofs of theorems and propositions as exercises if they are instructive and at least somewhat interesting.
- Each chapter features several trickier **problems** at the end. Some are reasonable, but others are legitimately difficult olympiad-style problems. Harder problems are marked with up to three chili peppers ( ), like this paragraph.

  In addition to difficulty annotations, the problems are also marked by how important they are to the big picture.

  - **Normal problems**, which are hopefully fun but non-central.
  - **Daggered problems**, which are (usually interesting) results that one should know, but won't be used directly later.
  - **Starred problems**, which are results which will be used later on in the book.[1]

Several hints and solutions can be found in Appendices B and C.

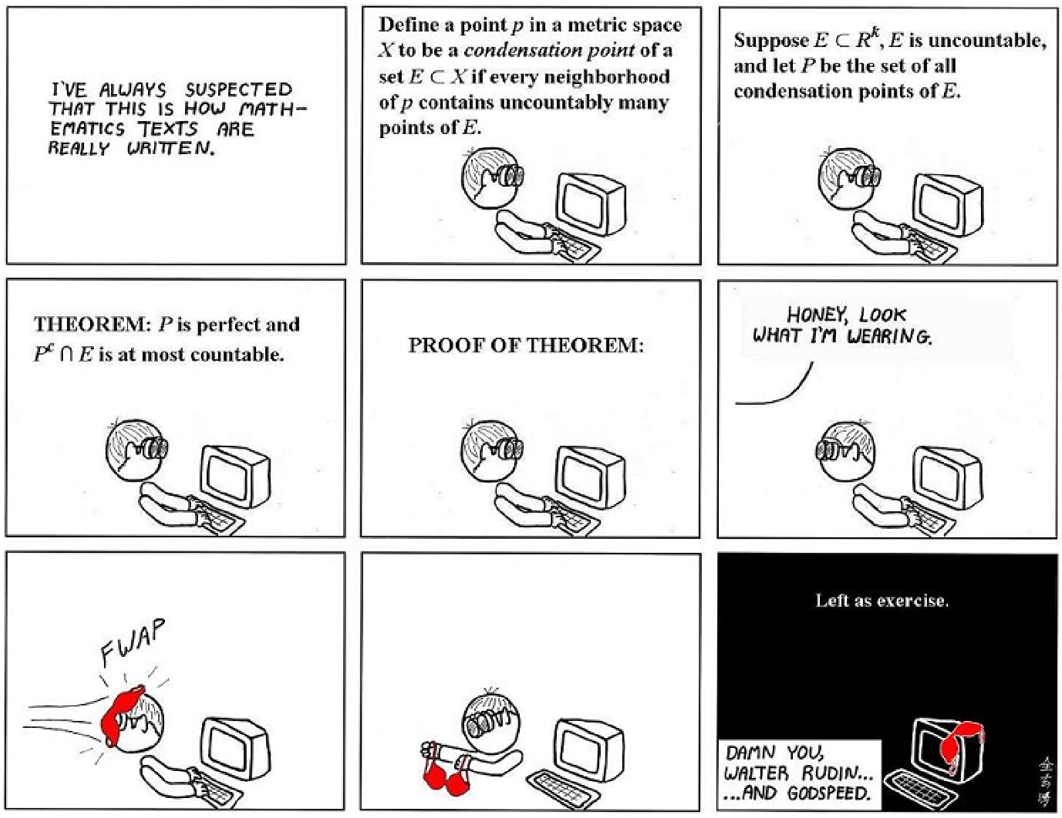


Image from [Go08]

[1]This is to avoid the classic "we are done by PSet 4, Problem 8" that happens in college sometimes, as if I remembered what that was.

§4 Paper

At the risk of being blunt,

Read this book with pencil and paper.

Here's why:

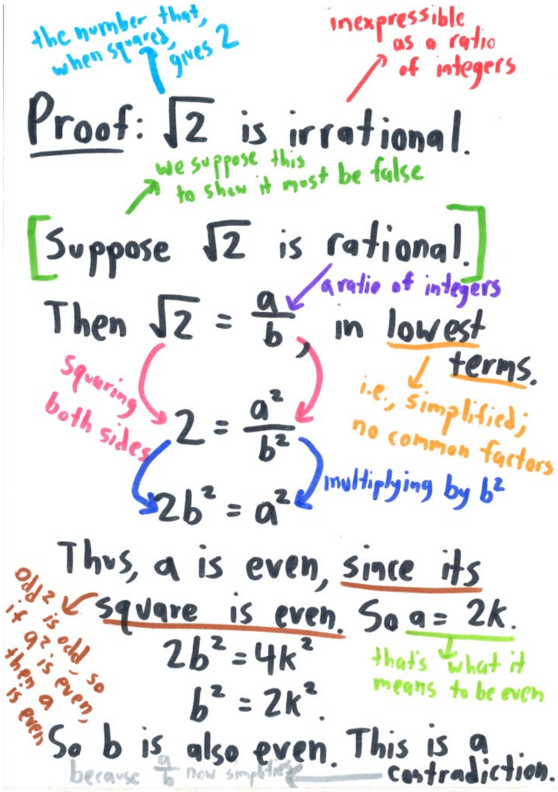


Image from [Or]

You are not God. You cannot keep everything in your head.[2] If you've printed out a hard copy, then write in the margins. If you're trying to save paper, grab a notebook or something along with the ride. Somehow, some way, make sure you can write. Thanks.

§5 On the importance of examples

I am pathologically obsessed with examples. In this book, I place all examples in large boxes to draw emphasis to them, which leads to some pages of the book simply consisting of sequences of boxes one after another. I hope the reader doesn't mind.

I also often highlight a "prototypical example" for some sections, and reserve the color red for such a note. The philosophy is that any time the reader sees a definition or a theorem about such an object, they should test it against the prototypical example. If the example is a good prototype, it should be immediately clear why this definition is intuitive, or why the theorem should be true, or why the theorem is interesting, et cetera.

Let me tell you a secret. Whenever I wrote a definition or a theorem in this book, I would have to recall the exact statement from my (quite poor) memory. So instead I often consider the prototypical example, and then only after that do I remember what the definition or the theorem is. Incidentally, this is also how I learned all the definitions in the first place. I hope you'll find it useful as well.

[2]See also https://blog.evanchen.cc/2015/03/14/writing/ and the source above.

§6 Conventions and notations

This part describes some of the less familiar notations and definitions and settles for once and for all some annoying issues ("is zero a natural number?"). Most of these are "remarks for experts": if something doesn't make sense, you probably don't have to worry about it for now.

A full glossary of notation used can be found in the appendix.

§6.i Natural numbers are positive

The set $\mathbb{N}$ is the set of *positive* integers, not including 0. In the set theory chapters, we use $\omega = \{0, 1, \dots\}$ instead, for consistency with the rest of the book.

§6.ii Sets and equivalence relations

This is brief, intended as a reminder for experts. Consult Appendix E for full details.

An **equivalence relation** on a set X is a relation $\sim$ which is symmetric, reflexive, and transitive. An equivalence relation partitions X into several **equivalence classes**. We will denote this by $X/\sim$. An element of such an equivalence class is a **representative** of that equivalence class.

I always use $\cong$ for an "isomorphism"-style relation (formally: a relation which is an isomorphism in a reasonable category). The only time $\simeq$ is used in the Napkin is for homotopic paths.

I generally use $\subseteq$ and $\subsetneq$ since these are non-ambiguous, unlike $\subset$. I only use $\subset$ on rare occasions in which equality obviously does not hold yet pointing it out would be distracting. For example, I write $\mathbb{Q} \subset \mathbb{R}$ since "$\mathbb{Q} \subsetneq \mathbb{R}$" is distracting.

I prefer $S \setminus T$ to $S - T$.

The power set of S (i.e., the set of subsets of S), is denoted either by 2^S or $\mathcal{P}(S)$.

§6.iii Functions

This is brief, intended as a reminder for experts. Consult Appendix E for full details.

Let $X \xrightarrow{f} Y$ be a function:

- By $f^{\text{pre}}(T)$ I mean the **pre-image**
$$f^{\text{pre}}(T) := \{x \in X \mid f(x) \in T\}.$$
This is in contrast to the $f^{-1}(T)$ used in the rest of the world; I only use f^{-1} for an inverse *function*.

  By abuse of notation, we may abbreviate $f^{\text{pre}}(\{y\})$ to $f^{\text{pre}}(y)$. We call $f^{\text{pre}}(y)$ a **fiber**.

- By $f^{\text{img}}(S)$ I mean the **image**
$$f^{\text{img}}(S) := \{f(x) \mid x \in S\}.$$
Almost everyone else in the world uses $f(S)$ (though $f[S]$ sees some use, and $f''(S)$ is often used in logic) but this is abuse of notation, and I prefer $f^{\text{img}}(S)$ for emphasis. This image notation is *not* standard.

- If $S \subseteq X$, then the **restriction** of f to S is denoted $f\restriction_S$, i.e. it is the function $f\restriction_S \colon S \to Y$.

- Sometimes functions $f \colon X \to Y$ are *injective* or *surjective*; I may emphasize this sometimes by writing $f \colon X \hookrightarrow Y$ or $f \colon X \twoheadrightarrow Y$, respectively.

§6.iv Cycle notation for permutations

Additionally, a permutation on a finite set may be denoted in *cycle notation*, as described in say https://en.wikipedia.org/wiki/Permutation#Cycle_notation. For example the notation $(1\ 2\ 3\ 4)(5\ 6\ 7)$ refers to the permutation with $1 \mapsto 2$, $2 \mapsto 3$, $3 \mapsto 4$, $4 \mapsto 1$, $5 \mapsto 6$, $6 \mapsto 7$, $7 \mapsto 5$. Usage of this notation will usually be obvious from context.

§6.v Rings

All rings have a multiplicative identity 1 unless otherwise specified. We allow $0 = 1$ in general rings but not in integral domains.

All rings are commutative unless otherwise specified. There is an elaborate scheme for naming rings which are not commutative, used only in the chapter on cohomology rings:

	Graded	Not Graded
1 not required	graded pseudo-ring	pseudo-ring
Anticommutative, 1 not required	anticommutative pseudo-ring	N/A
Has 1	graded ring	N/A
Anticommutative with 1	anticommutative ring	N/A
Commutative with 1	commutative graded ring	ring

On the other hand, an *algebra* always has 1, but it need not be commutative.

§6.vi Choice

We accept the Axiom of Choice, and use it freely.

§7 Further reading

The appendix Appendix A contains a list of resources I like, and explanations of pedagogical choices that I made for each chapter. I encourage you to check it out.

In particular, this is where you should go for further reading! There are some topics that should be covered in the Napkin, but are not, due to my own ignorance or laziness. The references provided in this appendix should hopefully help partially atone for my omissions.

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